Prerequisites

1) Basic algebraic geometry on affine / projective varieties.

e.g. Hartshorne chap I.

2) sheaves of modules and divisors

e.g. Hartshorne, §5, 6, 8

3) Cohomologies of sheaves : basic definitions and facts

e.g. Hartshorne, §2, 7

References

1) R. Lazarsfeld : Positivity in Algebraic Geometry I & II

2) S. Boucksom, J.P. Demailly, M. Paun and T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative kodaira dimension,

Conventions

1) $\mathbb{F} = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$

2) ~~vaste~~ For simplicity, all the varieties are defined over $\mathbb{C}$. However, most of the results hold for any alg. closed field k with $\mathrm{char}(k) = 0$

3) Varieties = reduced separated schemes of finite type / $\mathbb{C}$.

4) topology = Zariski topology unless otherwise stated.

Chapter I : Divisors, line Bundles and Intersection Numbers

§1. Weil divisors and Cartier divisors

Let X be an irreducible variety. $\mathbb{F} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$

A| Weil divisors

Def. -

1) A prime divisor D on X is a closed ~~sub~~ irreducible subvariety of codim one.

2) A Weil $\mathbb{F}$-divisor D on X is a formal finite sum $D = \sum a_i D_i$,

   where D_i's are prime divisors on X and $a_i \in \mathbb{F}$.

3) $WDiv_{\mathbb{F}}(X) = \{$ Weil $\mathbb{F}$-divisors on $X \}$.

Remark -

1) $WDiv_{\mathbb{F}}(X)$ is an abelian group. with natural additive structure.

2) $WDiv_{\mathbb{F}}(X)$ is a $\mathbb{F}$-vector space if $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$.

3) If $\mathbb{F} = \mathbb{Z}$, then usually we will omit $\mathbb{F}$ from the notations.

B| Cartier divisors.

Let X be an irreducible $\boxed{\text{normal}}$ variety. Then

a) $X_{sing} = \{x \in X \mid X \text{ is singular at } x\}$ is a closed set of X with codim ≥ 2.

b) [Hartogs' extension Thm].

   $\forall Z \subseteq X$ closed subset with codim ≥ 2. Then we have.

$$\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X \setminus Z, \mathcal{O}_X).$$

In particular, for any prime divisor D on X, a) implies ~~then~~ X is nonsingular at η_D where, η_D is the generic point of D, and $\mathcal{O}_{X,\eta_D}$ is a DVR.

Def. -

1) Let $0 \neq h \in K(X)$ be a rational function. Then we define.

$$\operatorname{div}(h) = \sum_{D \text{ prime}} \operatorname{ord}_D(h)\, D$$

   where $\operatorname{ord}_D(h)$ = valuation of h at η_D = multiplicity at h along D.

2) A principal divisor is a Weil divisor of the form $\mathrm{div}(h)$ for some $h \in K(X)$.

3) A Cartier divisor D is a Weil divisor which is locally principal, i.e. $\exists\, X = \bigcup U_i$ an open covering and $h_i \in K(U_i) = K(X)$ s.t. $D|_{U_i} = \mathrm{div}(h_i|_{U_i})$

4) A Cartier $\mathbb{F}$-divisor is a finite ~~sum of~~ formal sum ~~of~~ $\sum a_i D_i$ with D_i Cartier divisors and $a_i \in \mathbb{R}$.

may be not prime.

5) $\mathrm{Div}_{\mathbb{F}}(X) = \{$ Cartier $\mathbb{F}$-divisors on $X \}$.

Remark -

1) $\mathrm{Div}_{\mathbb{F}}(X)$ is an abelian group.

2) $\mathrm{Div}_{\mathbb{F}}(X)$ is a $\mathbb{F}$-vector space if $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$.

3) If $\mathbb{F} = \mathbb{Z}$, then we usually omit $\mathbb{F}$.

4) In general, $\mathrm{Div}_{\mathbb{F}}(X) \neq \mathrm{WDiv}_{\mathbb{F}}(X)$. However, if X is nonsingular, then every Weil divisor is Cartier and thus $\mathrm{Div}_{\mathbb{F}}(X) = \mathrm{WDiv}_{\mathbb{F}}(X)$.

c) Linear equivalence

Def. - Let X be an irreducible normal variety.

1) Two Weil $\mathbb{F}$-divisors D and D' are called $\mathbb{F}$-linearly equivalent if $\exists$ finitely many principal divisors $\mathrm{div}(h_i)$ and $r_i \in \mathbb{F}$ s.t.

$$D - D' = \sum r_i \,\mathrm{div}(h_i) \quad \Leftrightarrow \quad D \sim_{\mathbb{F}} D'$$

2) Let D be a Weil divisor on X. The divisorial sheaf $\mathcal{O}_X(D)$ associated to D is defined as:

$$U \longmapsto \Gamma(U, \mathcal{O}_X(D)) := \{ h \in K(U) \mid \mathrm{div}(h|_U) + D|_U \geq 0 \}.$$

where $U \subseteq X$ an open subset.

Remark -

1) Let D, D' be two Weil divisors. Then $D \sim D' \Leftrightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules

2) A Weil divisor D is Cartier iff $\mathcal{O}_X(D)$ is ~~locally free~~ an invertible sheaf.

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{h_i}\mathcal{O}_X|_{U_i} = \frac{1}{h_i}\mathcal{O}_{U_i}$$

§2. Line bundles

Let X be an irreducible variety.

A| Definition

Def. - A line bundle $L \overset{\text{over}}{\longrightarrow} X$ is a variety L with a surjective morphism $\pi : L \to X$ s.t. $\exists\, X = \bigcup U_i$ an open covering satisfying.

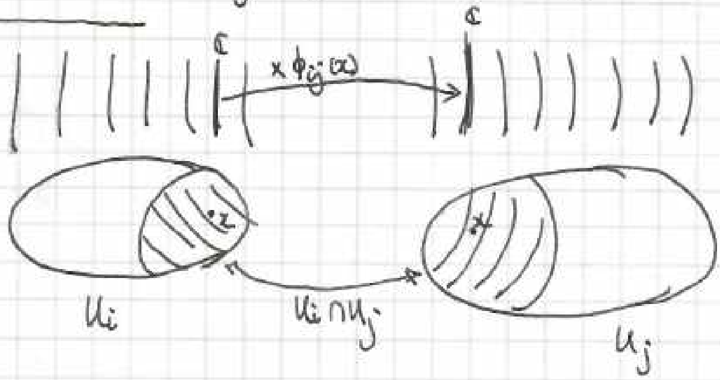
1) $$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C} \arrow[dl, "p_i"] \\ & U_i & \end{tikzcd}$$ p_i = first projection

2) $\forall\, i, j$,
$$\begin{tikzcd} U_i\cap U_j \times \mathbb{C} \arrow[rr, leftrightarrow, "\phi_j \circ \phi_i^{-1}"] \arrow[dr] & & U_i \cap U_j \times \mathbb{C} \arrow[dl] \\ & \pi^{-1}(U_i \cap U_j) \arrow[ul, "\phi_j"] \arrow[ur, "\phi_i"'] \arrow[d] & \\ & U_i \cap U_j & \end{tikzcd}$$

and $\phi_j \circ \phi_i^{-1}$ is pointwise linear isomorphism.

i.e. $\forall\, x \in U_i \cap U_j$, $\quad \phi_j \circ \phi_i^{-1}\big|_{\pi^{-1}(x)} : \mathbb{C} \longrightarrow \mathbb{C} \quad \in GL(\mathbb{C}) = \mathbb{C}^*$

$\rightsquigarrow \quad \phi_{ij} = \phi_j \circ \phi_i^{-1} : U_i \cap U_j \longrightarrow \mathbb{C}^*$ and $\pi^{-1}(x)$ carries a natural $\mathbb{C}$-vector space structure.



Remark -

1) A line bundle L is given by the following datas.

- $X = \bigcup U_i$ an open covering.
- $\phi_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^*$
- $\forall\, i, j, l$, we have $\phi_{lj} \cdot \phi_{il} = \phi_{ij}$

2) L^{-1} = dual bundle of L

$= (\bigcup U_i,\ \phi_{ij}^{-1})$

3) $L = (\bigcup U_i,\ \phi_{ij})$ and $L' = (\bigcup U_i,\ \phi'_{ij})$, then $L \otimes L' = (\bigcup U_i,\ \phi_{ij} \cdot \phi'_{ij})$.

4) $L \cong L'$ if $\exists$

$$\begin{tikzcd} L \arrow[rr, "\phi", "\cong"'] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & L' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

st $\forall x \in X$, $\phi\big|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \pi'^{-1}(x)$ is a linear isomorphism.

5) $\mathrm{Pic}(X) = \left(\{\text{line bundles}\}/\sim, \otimes, \mathbb{1} = X \times \mathbb{C} \right)$ is an abelian group.

B) From Cartier divisors to line bundles

Let $D = (\bigcup U_i, h_i)$ be a Cartier divisor on an irreducible normal variety.

Then define $\phi_{ij} = \dfrac{h_j}{h_i}\Big|_D : U_i \cap U_j \to \mathbb{C}^*$.

This is well-defined as $\mathrm{div}\left(h_j|_{U_i \cap U_j}\right) = \mathrm{div}\left(h_i|_{U_i \cap U_j}\right) = D|_{U_i \cap U_j}$.

Moreover, $\forall\, i, j, l$.

$$\phi_{lj} \cdot \phi_{il} = \frac{h_j}{h_l} \cdot \frac{h_l}{h_i} = \frac{h_j}{h_i} = \phi_{ij}$$

Hence, we can define $L_D = \left(\bigcup U_i, \phi_{ij} = \dfrac{h_j}{h_i}\right)$.

Prop.- Let X be an irreducible normal variety.

$(\mathrm{Div}(X)/\sim, +) \longrightarrow \mathrm{Pic}(X)$ is ~~an~~ ~~isom~~ a homomorphism of abelian gps

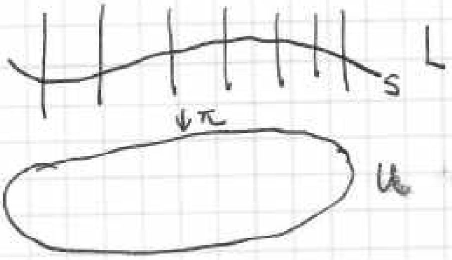
$$[D] \longmapsto [L_D]$$

Moreover, it is actually an isomorphism.

C) sheaf of sections

Let X be an irreducible variety. $L \xrightarrow{\pi} X$ a line bundle. The sheaf $\mathcal{L}$ of sections of L is defined as

$$U \longmapsto \Gamma(U, \mathcal{L}) := \{ s : U \to L \mid \pi \circ s = \mathrm{Id}_U \}$$



Then $\mathcal{L}$ is an invertible sheaf as $L|_{U_i} \cong U_i \times \mathbb{C}$ and $\mathcal{L}|_{U_i} = \mathcal{O}_X|_{U_i} = \mathcal{O}_{U_i}$

Prop.- Let X be an irreducible normal variety. D a Cartier divisor on X. $L_D \to X$ the associated line bundle. $\mathcal{L}_D$ the sheaf of sections of L_D. Then

$$\mathcal{O}_X(D) \cong \mathcal{L}_D \quad \text{as } \mathcal{O}_X\text{-modules}.$$

Proof.- Assume $D = (\bigcup U_i, h_i)$, where $h_i \in K(X)$ s.t. $\mathrm{div}(h_i|_{U_i}) = D|_{U_i}$

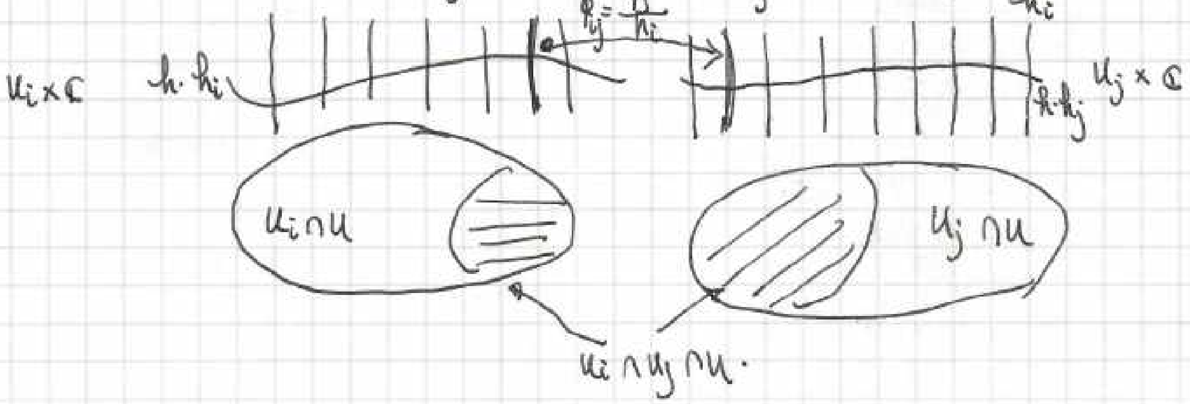
Let $U \subseteq X$ be an arbitrary open subset. Recall that we have.

$$\Gamma(U, \mathcal{O}_X(D)) = \{ h \in K(X) \mid \mathrm{div}(h|_U) + D|_U \geq 0 \}.$$

Thus, for $\forall i$, $h \cdot h_i \in \Gamma(U \cap U_i, \mathcal{O}_X)$.

Define
$$\Phi_U : \Gamma(U, \mathcal{O}_X(D)) \longrightarrow \Gamma(U, \mathcal{L}_D)$$
$$h \longmapsto (h \cdot h_i)_i$$

This is well-defined as $h \cdot h_j = h \cdot h_i \cdot \frac{h_j}{h_i} = h \cdot h_i \cdot \phi_{ij}$



$$\Phi_U^{-1} : \Gamma(U, \mathcal{L}_D) \longrightarrow \Gamma(U, \mathcal{O}_X(D))$$
$$s \longmapsto \frac{\boxed{s|_{U_i}}}{h_i} \qquad \boxed{s|_{U_i}} \in \Gamma(U_i \cap U, \mathcal{O}_X), \quad \frac{s|_{U_i}}{h_i} \in K(U \cap U_i) = K(X).$$

This is well-defined as
$$\frac{s|_{U_j}}{h_j} = \frac{s|_{U_i} \cdot \frac{h_j}{h_i}}{h_j} = \frac{s|_{U_i}}{h_j} \quad \text{in } K(X).$$

D) From line bundles to Cartier divisors

Def.- Let X be an irreducible variety. $L \to X$ a line bundle given by $(\bigcup U_i, \phi_{ij})$. Then a rational section s of L is a collection (s_i) s.t.

1) $s_i \in K(U_i) = K(X)$
2) $s_j = \phi_{ij} s_i \in K(U_j) = K(X)$

Remark -

1) {rational sections of L} = $K(X)$.

$$s \longmapsto s_i \quad (\forall \text{ fixed } i)$$

$$\underbrace{\left.\begin{array}{l} s_i = h \\ s_j = h\cdot\phi_{ij} \end{array}\right\}}_{} \quad (s_j) \longleftarrow h \quad (\forall \text{ fixed } i).$$

well-defined $s_j = h\cdot\phi_{ij} = h\cdot\phi_{il}\cdot\phi_{lj}$

$= s_l\cdot\phi_{lj}$ in $K(X)$.

2) $\Gamma(X,\mathcal{L})$ = {rational sections $s=(s_i)$ | $s_i\in\Gamma(U_i,\mathcal{O}_X)$}.

Let X be an irreducible normal variety. s a rational section of L.

Define $\operatorname{div}(s)$ to be the unique Weil divisor D s.t.

$$D|_{U_i} = \operatorname{div}(s_i|_{U_i}), \quad \forall i.$$

well-defined as $s_j|_{U_i\cap U_j} = \underbrace{\phi_{ij}}_{\in\,\mathcal{O}^*_{U_i\cap U_j}}\, s_i|_{U_i\cap U_j}$

Hence, $\operatorname{div}(s)$ is a Weil divisor.

Prop. - Let $L\to X$ be a line bundle over an irreducible normal variety.

Let $s=(s_i)$ and $s'=(s'_i)$ be two rational sections of L.

then 1) $\operatorname{div}(s) \sim \operatorname{div}(s')$.

2) $\operatorname{div}(s)\geq 0 \iff s$ is a global section of L

Proof. - Define $h = \frac{s_j}{s'_j} = \frac{\phi_{ij}\, s_i}{\phi_{ij}\cdot s'_i} = \frac{s_i}{s'_i} \in K(X)$.

and $\operatorname{div}(h) = \operatorname{div}(s) - \operatorname{div}(s')$.

Prop. - Let X be an irreducible normal variety.

$$\operatorname{Div}(X)/\sim \;\xrightarrow{\cong}\; \operatorname{Pic}(X) \quad \text{as groups}.$$

$$[D] \longmapsto [L_D]$$

$$[\operatorname{div}(s)] \longleftarrow\!\!\shortmid\; 0\neq s = \text{rational section of } L.$$

§3. Linear system.

A) Definition

~~Let L → X be one line bundle over~~

Let $L \longrightarrow X$ be a line bundle over an ~~irreducible~~ variety X.

Let $V \subseteq \Gamma(X, \mathcal{L})$ be a finite dimensional linear subspace. We define.

$$\begin{array}{ccccc} \Phi_V : & X & \dashrightarrow & \mathbb{P}(V) := \mathbb{P}_{sub}(V^*) = V\setminus\{0\} \big/ \mathbb{C}^* \\ & \Psi & & \Psi \\ & x & \longmapsto & [H_x] = [H_x^{\perp}] \end{array}$$

codim one linear subspace

one dim'l linear subspace

where $H_x = \{ s \in V \mid s(x) = 0 \}$. In particular, Φ_V is well-defined at x if $\exists\ s \in V$ s.t. $s(x) \neq 0$.

Def. Let X be an irreducible normal variety. D a Cartier divisor on X.

1) The complete linear system $|D|$ is defined to be one of the following.

a) $\{ D' \text{ Cartier divisor} \mid D' \geq 0,\ D' \sim D \}$.

b) $\{ \operatorname{div}(s) \mid s \in \Gamma(X, \mathcal{L}_D) \}$

c) $\{ \operatorname{div}(h) + D \mid h \in \underbrace{\Gamma(X, \mathcal{O}_X(D))}_{\shortparallel} \}$.

$\{ h \in K(X) \mid \operatorname{div}(h) + D \geq 0 \}$.

2) A linear system $|V| \subseteq |D|$ is the image of a linear subspace V of $\Gamma(X, \mathcal{L}_D)$ under the map

$$\begin{array}{ccc} \Gamma(X, \mathcal{L}_D) & \longrightarrow & |D| \\ \cup\!\mid & & \cup\!\mid \\ \tilde{V} & \longrightarrow & |V| \end{array}$$

Remark 1) ~~and~~ $|V| = \mathbb{P}_{sub}(\tilde{V})$ as sets. and hence $|V|^* := \mathbb{P}_{sub}(\tilde{V}^*) = \mathbb{P}(\tilde{V})$

$\rightsquigarrow$ $\Phi_V := \Phi_{\tilde{V}} : X \dashrightarrow |V|^* = \mathbb{P}^N$, $N = \dim_{\mathbb{C}} V$

If X is projective/complete

$\operatorname{div}(s) = \operatorname{div}(s') \Rightarrow \frac{s}{s'} \in \Gamma(X, \mathcal{O}_X) = \mathbb{C} \Rightarrow \frac{s}{s'}$

X complete.

2) Given a linear system, then we define

$$\mathrm{Bs}\,|V| = \{ x \in X \mid s(x) = 0, \ \forall \ s \in V \}.$$

$$= \{ x \in X \mid x \in \mathrm{Supp}(\mathrm{div}(s)), \ \forall \ s \in V \}$$

Then $\mathrm{Bs}|V|$ is a closed subset of X. s.t. $\Phi_{|V|}$ is not well-defined

B| Tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$

Let V be a vector space of rank $n+1$.

$$\mathbb{P}^n := \mathbb{P}(V) = \{ \text{codim } 1 \text{ linear subspaces of } V \} \qquad [H]$$

$$= \{ \text{one dimensional linear subspaces of } V^* \}. \qquad [H^\perp].$$

$$=: \mathbb{P}_{\mathrm{sub}}(V).$$

$$\begin{array}{ccccccccc} & & \text{universal bundle} & & & & \text{tautological bundle} & & \\ 0 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathbb{P}^n \times V & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1) & \longrightarrow & 0. \\ & & \parallel & & & & \downarrow \pi & & \\ & & \{([H], v) \mid v \in H\} & & & & \mathbb{P}^n & & \end{array}$$

Thus $\pi^{-1}([H]) \cong V/H$. which is one-dimensional, hence $\mathcal{O}_{\mathbb{P}^n}(1)$ is a line bundle, which is called the tautological line bundle of $\mathbb{P}^n$.

Remark - (Datas for $\mathcal{O}_{\mathbb{P}^n}(1)$)

1) Let $[x_0 : \cdots : x_n]$ be the standard homogeneous coordinates on $\mathbb{P}^n$.
Let $U_i = \{x_i \neq 0\} \cong \mathbb{C}^n$ be the standard open covering of $\mathbb{P}^n$.

Then $\Gamma(U_i, \mathcal{O}_X) = \mathbb{C}[\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \cdots \frac{x_n}{x_i}]$.

$$\rightsquigarrow \quad \phi_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^*$$

$$[x_0 : \cdots : x_n] \longmapsto \frac{x_i}{x_j}$$

2) $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = \{ \underbrace{\sum_{i=0}^n a_i x_i}_{=: s} \mid a_i \in \mathbb{C} \}$ in the following sense,

$$s_i := s|_{U_i} = \frac{s}{x_i}\Big|_{U_i} \in \Gamma(U_i, \mathcal{O}_X). \quad \rightsquigarrow \quad s_j = \frac{s}{x_j} = \frac{s}{x_i} \cdot \frac{x_i}{x_j} = s_i \cdot \phi_{ij}.$$

3) ~~For $m \in \mathbb{Z}$,~~ $\mathcal{O}_{\mathbb{P}^n}(-1)$ = dual bundle of $\mathcal{O}_{\mathbb{P}^n}(1)$.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathbb{P}^n \times V^* \longrightarrow U^* \longrightarrow 0.$$

$$\| $$

$$\{([H_x^\perp], v) \mid v \in H_x^\perp \subseteq \mathbb{P}_{sub}(V^*) = \mathbb{P}(V)\}.$$

$$\rightsquigarrow \quad \phi_{ij}' = \phi_{ij}^{-1} = \frac{X_j}{X_i}$$

4) $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \begin{cases} S_m = \text{homogeneous polynomials of deg. } m, & m \geq 0 \\ 0 & m < 0 \end{cases}$

c) Embedding and $\mathcal{O}_{\mathbb{P}^n}(1)$.

1) Let $X \subseteq \mathbb{P}^n$ be an irreducible quasi-projective variety. Let $L := \mathcal{O}_{\mathbb{P}^n}(1)|_X$.

Let $V := \mathrm{Im}(\Gamma(X, \mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow \Gamma(X, L))$.

Then $\Phi_V : X \longrightarrow X \subseteq \mathbb{P}^n$ is the identity map.

2) Conversely, let $L \to X$ be a line bundle over an irreducible variety X.

Let $V \subseteq \Gamma(X, \mathcal{L})$ be a finite dimensional subspace s.t $Bs|V| = \phi$.

Then $\Phi_V : X \longrightarrow \mathbb{P}(V)$ is a morphism s.t. $L \cong \Phi_V^* \mathcal{O}_{\mathbb{P}(V)}(1)$.

$\rightsquigarrow$ The main body of this course is to understand those L ~~&~~. or D

s.t. $Bs|D| = \phi$ or $\Phi_{|D|}$ is an embedding.

§4. Intersection numbers

A) Degree functions

Let C be a nonsingular projective curve. $D = \sum a_i P_i$ be a Cartier $\mathbb{F}$-divisor on C, where $P_i \in C$ are points and $a_i \in \mathbb{F}$. The <u>degree function</u> is defined as

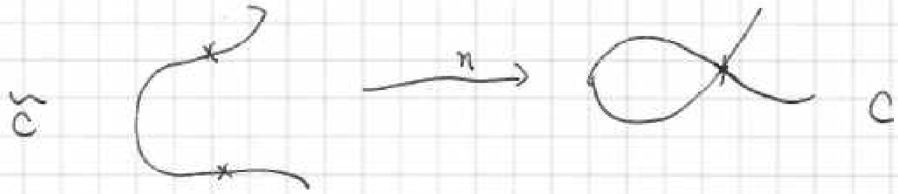
$$\deg(D) = \underbrace{\sum a_i}_{\text{finite sum.}} \in \mathbb{F}$$

<u>Prop.</u> - [Hartshorne, II, Prop. 6.9 & Cor. 6.10]

If D is a principal divisor on C, then $\deg(D) = 0$.

$\Rightarrow$ In particular, if $D \sim_{\mathbb{F}} D'$, then $\deg(D) = \deg(D')$.

<u>Recall</u> - Let C be an irreducible proj. curve. $n : \tilde{C} \longrightarrow C$ be the normalisation. Then $\tilde{C}$ is nonsingular because $\tilde{C}_{sing}$ has codim ≥ 2.



<u>Def.</u> - Let L be a line bundle over an irreducible projective variety X. Let $C \subseteq X$ be an irreducible proj. curve. Then

$$L \cdot C := \deg(n^* L) \in \mathbb{Z}.$$

n^*L: line bundle over $\tilde{C}$ $\rightsquigarrow$ Cartier divisor on $\tilde{C}$

<u>Examples</u>

1) Let $f : X \longrightarrow Y$ be a morphism between irreducible projective varieties. Let $L \rightarrow Y$ be a line bundle and $C \subseteq X$ an irreducible projective curve st. $f(C) = p \in Y$ is a pt. Then $f^*L \cdot C = 0$.

Indeed, let $p \in \mathcal{U}$ be an open nbhd of p s.t. $L|_{\mathcal{U}} = \mathcal{U} \times \mathbb{C}$ is trivial.

Thus $f^*L|_{f^{-1}(\mathcal{U})}$ is trivial and hence $\nu^* f^* L$ is trivial

$$\Rightarrow \quad \deg(\nu^* f^* L) \overset{\deg}{=} \operatorname{div}(1) = 0$$

2) Let $L \to X$ be a line bundle over an irreducible proj. var. X. $C \subseteq X$ an irred. proj. curve. Assume $\exists\, s \in \Gamma(X, \mathcal{L})$ s.t. $s|_C \not\equiv 0$. Then $L \cdot C \geq 0$ with equality iff $s|_C$ is nowhere vanishing.

Indeed, $0 \neq \nu^* s \in \Gamma(\tilde{C}, \nu^* L) \Rightarrow \operatorname{div}(\nu^* s) \geq 0$

$\Rightarrow \deg(\nu^* L) \geq 0$

Moreover, if $\deg(\nu^* L) = 0$, then $\operatorname{div}(\nu^* s) = 0$ i.e. $\nu^* s$ has no zeros.

Def. - Let X be an irreducible normal projective variety. $D \underset{= \Sigma a_i D_i}{}$ a Cartier $\mathbb{F}$-divisor. $C = \Sigma b_j C_j$ formal finite sum with $C_i \subseteq X$ irreducible projective curves with $a_i \in \mathbb{F}$. Define the <u>intersection number</u> $D \cdot C$ as

$$D \cdot C = \sum_{i,j} a_i b_j \, L_{D_i} \cdot C_j \in \mathbb{F}.$$

<u>Remark</u> -

1) If $D \sim_{\mathbb{F}} D'$, then $D \cdot C = D' \cdot C$.

2) If $D \geq 0$ and $C \subseteq X$ irreducible proj. curve s.t. $C \not\subseteq \operatorname{Supp}(D)$, then $D \cdot C \geq 0$ with equality iff $\operatorname{Supp}(D) \cap C = \phi$.

3) If $\operatorname{Supp}(D) \cap C = \phi$, then $D \cdot C = 0$.

B) <u>Riemann-Roch thm</u>

<u>Thm</u> - Let C be a nonsingular irreducible projective curve. D a Cartier divisor. Then

$$\chi(C, \mathcal{O}_C(D)) := h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = \deg(D) + 1 - g.$$

where $h^i(C, \mathcal{O}_C(D)) = \dim_{\mathbb{C}} H^i(C, \mathcal{O}_C(D))$ and $g = H^0(C, \omega_C)$ is the genus of C, $\omega_C = \Omega^1_C$ is the differential sheaf of C.

Remark -

1) If $\deg(D) > g-1$, then $h^0(C, \mathcal{O}_C(D)) > 0$, i.e. $\Gamma(C, \mathcal{O}_C(D)) \neq \phi$.

2) Let $P(t) = \deg(D)t + 1 - g$. Then for any $n \in \mathbb{Z}$, we have

$$\chi(C, \mathcal{O}_C(nD)) = P(n).$$

$\rightsquigarrow$ $L_D \cdot C$ = coefficient of "highest ~~term~~ degree term of $P(n)$".

C] General definition of intersection numbers.

Let X be an irreducible projective variety of dimension $n \geq 1$. Let $L_1, \dots, L_r$ be line bundles ~~of~~ on X with $r \geq n$. Then $\exists$ a polynomial $P \in \mathbb{Q}[T_1, \dots, T_r]$ s.t. $\deg(P) \leq n$ and for any $(m_1, \dots, m_r) \in \mathbb{Z}^r$, we have

$$\chi(X, L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}) := \sum_{i=0}^{n} (-1)^i h^i(X, L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}).$$

$$\| $$

$$P(m_1, \dots, m_r).$$

Def. -

$L_1 \cdots L_r :=$ coefficient of $T_1 \cdots T_r$ in P.

~~Remar~~

Properties of intersection numbers

1) If $r > n$, the $L_1 \cdots L_r = 0$ as $\deg(P) \leq n$.

~~2) ...~~

2) If X is an irreducible normal projective variety, the $L_1 \cdots L_n$ can be extended to Cartier $\mathbb{F}$-divisors such that

$$(D_1, \dots, D_n) \longmapsto D_1 \cdots D_n \in \mathbb{F}$$

is multilinear, symmetric and independent of ~~linear~~ representative in $\mathbb{F}$-linear equivalent class.

3) [Projection formula]

~~Recall - (Push-forward of cycles):~~

Let $f: X \to Y$ be a ~~dominant~~ (surj.) morphism between irreducible projective varieties. ~~Then~~, $L_1, \dots, L_r$ line bundles on Y s.t. $r \geq \dim(X) = n$. Then

$$f^*L_1 \cdot \dots \cdot f^*L_r = \deg(f)\, L_1 \cdots L_r$$

where
$$\deg f = \begin{cases} 0 & \text{if } \dim X > \dim Y \\ \\ ~~\deg~~ [K(X) : K(Y)] & \text{if } \dim X = \dim Y. \end{cases}$$

In particular, we note that the normalisation of a variety is finite, thus for intersection numbers, it is harmless to restrict ourselves to <u>irreducible normal proj. var.</u>

and the definition of $L \cdot C$ coincides with the definition using degree function. norm. map $Y \subseteq X$ irred. subvar. $D_1, \dots, D_r$ Cartier $\mathbb{F}$-div, $D_1 \cdot \dots \cdot D_r|_Y := n^*D_1 \cdots n^*D_r$ where $n: \tilde{Y} \to Y \to X$

4) Let $D_1, \dots, D_n$ be prime divisors on a nonsingular projective variety ~~that~~ meet transversely, ~~at smooth pt~~ then

$$D_1 \cdots D_n = \#\{D_1 \cap \cdots \cap D_n\}.$$

5) [Chow's moving lemma]

Let X be a nonsingular irreducible proj. variety of dim n. Let $D_1, \dots, D_n$ be Cartier divisors on X. Then for $\forall i$, $\exists\, D_i \sim \sum_{j=1}^{r_i} a_{ij} D_{ij}$ s.t. for any $j_1, \dots, j_n$, the divisors $D_{1j_1}, \dots, D_{nj_n}$ meet transversely and hence

$$D_{1j_1} \cdots D_{nj_n} = \#\{D_{1j_1} \cap \cdots \cap D_{nj_n}\}$$

<u>Remark</u> -

Let $D_1, \dots, D_n$ be ~~an~~ effective Cartier $\mathbb{F}$-divisors on a ~~an irred. normal~~ nonsing. proj. var. ~~proj.~~ variety X. Then $D_1 \cdots D_n$ may be negative.

Indeed, by chow's moving lemma, $D_i \sim \sum a_{ij} D_{ij}$. However, a_{ij} may be not positive furthermore! This means that you can not deform D_i effectively.

6) [Restriction] $D_1, \dots, D_{n-1}$ Cartier $\mathbb{F}$

Let ~~D~~ $\Delta = \sum a_i \Delta_i$ be a ~~prime~~ $\mathbb{F}$ Cartier divisor on an irred. normal proj. variety, ~~s.t. ... Cartier~~ $D_1, \dots, D_{n-1}$ Cartier $\mathbb{F}$-divisors on X. Then $D_1 \cdots D_{n-1} \cdot \Delta = \sum a_i (D_1|_{\Delta_i} \cdots D_{n-1}|_{\Delta_i})$

§5. Geometric and numerical properties

A) Geometric notion of positivity.

Def. - (for ~~Cartier divisors~~ line bundles).

Let X be ~~an irreducible normal~~ a projective variety. $L \to X$ a line bundle ~~s.t. $L = L_D$, for some where D a Cartier divisors on X~~.

1) L is <u>very ample</u> if $\Phi_{|L|} : X \longrightarrow |L|^* = \mathbb{P}(\Gamma(X,L))$ is an embedding.
2) L is <u>ample</u> if $L^{\otimes m}$ is very ample for some $m \in \mathbb{Z}_{>0}$
3) L is <u>globally generated</u> if $\mathrm{Bs}|L| = \phi$.
4) L is <u>semi-ample</u> if $L^{\otimes m}$ is globally generated for some $m \in \mathbb{Z}_{>0}$
5) L is <u>big</u> if $\Phi_{|L^{\otimes m}|} : X \dashrightarrow Y \subseteq |L|^*$ is birational for some $m \in \mathbb{Z}_{>0}$.

$$\begin{tikzcd} & & \text{big} \\ \text{very ample} \arrow[r, Rightarrow] & \text{ample} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & & \text{globally generated} \arrow[r, Rightarrow] & \text{semiample.} \end{tikzcd}$$

Def. - Let X be an irreducible normal projective normal variety.

1) A Cartier divisor D on X is $\mathcal{P}$ if so is L_D.
   $\mathcal{P} \in \{\text{very ample}, \cdots, \text{big}\}$.
2) A Cartier $\mathbb{F}$-divisor D on X is $\mathcal{P}^{\in\{\text{ample, big}\}}$ if $\exists\, D' \sim_{\mathbb{F}} D$ s.t. $D' = \sum a_i D_i'$ with $a_i \in \mathbb{F}_{>0}$ and D_i''s are Cartier divisors with $\mathcal{P}$.

Remark - (positivity vs intersection numbers : a first glimpse).

1) L ample, $C \subseteq X$ an irreducible projective curve. Then $L \cdot C > 0$.
   Indeed, we may assume L is very ample, and fix a pt $p \in C$.
   Then $\Phi_{|L|} : X \longrightarrow \mathbb{P}^N$ is an embedding s.t. $L = \Phi_{|L|}^* \mathcal{O}_{\mathbb{P}^N}(1)$.
   Let $s \in \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ be a linear form s.t. $\mathrm{div}(s) = H_s \subseteq \mathbb{P}^N$ is a hyperplane s.t. $p \in H_s$ and $C \not\subseteq H_s$. Hence,
   $$L \cdot C = \Phi_{|L|}^* H_s \cdot C > 0$$

2) L semi-ample and $C \subseteq X$ an irreducible projective curve. Then $L \cdot C \geqslant 0$. Moreover, if $L \cdot C = 0$, then $\Phi_{|L^{\otimes m}|}(C) = pt$ for $m \in \mathbb{Z}_{>0}$ s.t. $Bs|L^{\otimes m}| = \phi$.

Indeed, we may assume that L is globally generated, i.e. $Bs|L| = \phi$. Then $\Phi_{|L|} : X \longrightarrow \mathbb{P}^N$ is everywhere well-defined. Let $p \in C$ be a fixed pt. As $Bs|L| = \phi$, thus $\exists\, D \in |L|$ s.t. $p \notin D$ and hence $D \cdot C \geqslant 0$. Moreover, if $\Phi_{|L|}(C) \neq pt$, as in 1), we can find $H_s \in |\mathcal{O}_{\mathbb{P}^N}(1)|$ s.t. $p \in H_s$ and $C \not\subseteq H_s$ and hence $L \cdot C > 0$.

3) L is big, then $\exists\, Z \subseteq X$ proper closed subset s.t. $L \cdot C > 0$ for any irred. proj. curve $C \not\subseteq Z$.

Indeed, we may assume that $\Phi_{|L|} : X \dashrightarrow Y \subseteq \mathbb{P}^N$ is biration.

$$\begin{array}{ccc} \text{open} \cup\!| & & \cup\!|\text{open} \\ U & \xrightarrow{\ \simeq\ } & V \end{array}$$

Set $Z = X \setminus U$. Then as in 1), for any $C \not\subseteq Z$ irreducible proj. curve we have always $L \cdot C > 0$.

Question -

How about the converse? This is the question we want to answer in this course.

B) Numerical equivalence

Def. - Let X be an irreducible normal projective variety.

1) Let D and D' be two Cartier $\mathbb{F}$-divisors. We say that D and D' are <u>numerically equivalent</u>, i.e. $D \equiv_{\mathbb{F}} D'$, if for any irreducible proj. curve $C \subseteq X$, we have $D \cdot C = D' \cdot C$.

$\rightsquigarrow D \cdot C = D' \cdot C$ for any one $\mathbb{F}$-cycle.

2) Let C and C' be two one $\mathbb{F}$-cycles. We say that C and C' are <u>numerically equivalent</u>, i.e. $C \equiv_{\mathbb{F}} C'$ if for any Cartier divisor D we have $D \cdot C = D \cdot C'$.

Notation.

1) $N^1(X)_{\mathbb{F}} = \{\text{Cartier } \mathbb{F}\text{-divisors}\}/\equiv_{\mathbb{F}}$ is a free $\mathbb{Z}$-module of finite rank.

2) $N_1(X)_{\mathbb{F}} = \{\text{one } \mathbb{F}\text{-cycles}\}/\equiv_{\mathbb{F}}$ - - - - - - - - - - -

The pairing $N^1(X)_{\mathbb{F}} \times N_1(X)_{\mathbb{F}} \longrightarrow \mathbb{F}$ is bilinear and non-deg.

$$([D] \ , \ [C]) \longmapsto D\cdot C$$

Def - The Picard number $\rho(X)$ of X is defined as

$$\rho(X) = \mathrm{rk}\, N^1(X)_{\mathbb{F}} = \mathrm{rk}\, N_1(X)_{\mathbb{F}}.$$

Question -

which geometric properties of D are numerical?

Answer: ample and big. (TBD)

~~Lemma -~~

~~Let $D_1, \ldots, D_n$ ($D_1', \ldots, D_n'$) be Cartier $\mathbb{F}$-divisor on an irreducible (normal) projective variety X.~~

~~Let D_i's be Cartier s.t. $D_i \equiv_{\mathbb{F}} D_i'$.~~

~~Then $D_1 \cdots D_n = D_1' \cdots D_n'$.~~

Chapter II - Ampleness and nefness.

§1 - Hironaka's Thm and Bertini Thm.

A) Hironaka's Thm.

1) Let X be an irreducible ~~proj~~ variety. Then $\exists\ \hat{X} \longrightarrow X$ a birational projective morphism s.t. $\hat{X}$ is nonsingular.

2) Let $f: X \dashrightarrow Y$ be a rational map between irreducible varieties. Then $\exists\ \hat{X} \xrightarrow{\pi} X$ a birational projective morphism and $p: \hat{X} \longrightarrow Y$ s.t. $\hat{X}$ is nonsingular and

$$\begin{tikzcd} & \hat{X} \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ X \arrow[rr, dashed, "f"'] \arrow[rr, phantom, shift left=4, "\circlearrowright"] & & Y \end{tikzcd}$$

$\rightsquigarrow$ Hence, for intersection numbers, it is enough to consider nonsing. proj. var.

$\Rightarrow D_1 \equiv D_1' \Rightarrow D_1 \cdots D_n = D_1' \cdots D_n'$.

$\Rightarrow$ if $D \equiv 0$ ~~then~~ then $D^{\dim(Y)} \cdot Y = 0$, $\forall Y \subseteq X$ irred. ....

B) Bertini's Thm

Let X be a nonsingular variety. $x \in X$ a point. $D \geq 0$ an effective Cartier divisor.

$$\mathrm{mult}_x D = \mathrm{mult}_x (h_x),$$

where $h \in K(X)$ s.t. $\exists\ x \in U$ an open nbhd s.t. $\mathrm{div}(h|_U) = D|_U$.

Given a linear system $V \subseteq |D|$, we define

$$\mathrm{mult}_x V = \inf_{D' \in V} \mathrm{mult}_x D'.$$

Thm - (Bertini's Thm)

Let X be an irreducible nonsingular ~~pro~~ variety. D a Cartier divisor on X. $|V| \subseteq |D|$ a finite dimensional linear system. Then for any general $D' \in |V|$, we have: for any $x \in X$.

$$\mathrm{mult}_x D \leq \mathrm{mult}_x V + 1.$$

In particular, if $\mathrm{Bs}|V| = \phi$, then D' is nonsingular.

Remark -

3) + see latter

1) "general" means "not ~~cont~~ contained in some fixed proper closed subset"
$\Rightarrow |V| \cong \mathbb{P}_{\mathrm{sub}}(V)$ is a projective space with Zariski topology.

– (Nakai's ineq)

Cor. Let X be an irreducible projective variety. L_i ample line bundle on X. Then

$$\cancel{L^{\dim X} > 0}. \quad L_1 \cdots L_n > 0.$$

Proof. – WLOG, we may assume that ~~Q~~, L is very ample. Let $f : \hat{X} \to X$ be a resolution of X. Then by projection formula, we have

$$L^{\dim X} = (f^*L)^{\dim \hat{X}} = (f^*L)^{\dim X}$$

Pass to resolution and use Bertini's Thm. + restriction to reduce it to curve.

$n = \dim X = \dim \hat{X}$.

Let $V = \mathrm{Im}(\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \longrightarrow \Gamma(\hat{X}, f^*L))$.

$$\Phi_{|V|} : \hat{X} \longrightarrow X \xrightarrow{\Phi_{|L|}} \mathbb{P}^N$$

By Bertini's Thm, we may choose general $D_1, \dots, D_n \in |V|$ s.t. $\forall i$ $D_{i+1}\big|_{D_1 \cap \cdots \cap D_i}$ is nonsingular.

i.e. $D_1, \dots, D_n$ ~~is~~ meet ~~p~~ transversely and hence

$$(f^*L)^n = D_1 \cdots D_n = \# \{D_1 \cdots D_n\}.$$

Note that $\underset{\substack{\| \\ C}}{\dim(D_1 \cap \cdots D_{n-1})} = 1$ and $\Phi_{|V|}(C) \neq$ pts because $\Phi_{|V|}$ is birational and D_i's are general. Thus $D_n \cdot C > 0$

Remark –

3) Let $D \geq 0$ be an effective Cartier divisor on an irreducible ~~diviso~~ variety X. Then $\mathcal{O}_X(-D)$ is actually an ideal sheaf of $\mathcal{O}_X$, denoted by $\mathcal{I}_D$. In Bertini's Thm, ~~not only~~ D is actually nonsingular as the closed subscheme of X defined by $\mathcal{O}_X / \mathcal{I}_D$.

§2. Nakai - Moishezon - Kleiman Thm.

A| Recalling : basic facts about ampleness

Exercise

1) L very ample and L' globally generated $\Rightarrow$ $L \otimes L'$ very ample
2) L ample and L' semi-ample $\Rightarrow$ $L \otimes L'$ ample.
3) L globally generated $\Rightarrow$ $\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t. $L^{\otimes n}$ is g.g. for $n \geq n_0$.
4) L ample $\Rightarrow$ $\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t. $L^{\otimes n}$ is very ample for $n \geq n_0$.
5) L ample, L' arbitrary $\Rightarrow$ $\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t. $L^{\otimes n} \otimes L'$ is very ample for any $n \geq n_0$.

B| Cohomologies of ~~ample~~ line bundles.

Thm - Let $L \to X$ be an ~~ample~~ line bundle over a ~~[illegible]~~ proj. var. X.

~~1) [Serre vanishing]~~ 1) [Growth of Cohomologies] $h^i(X, \mathcal{F}(mD)) = O(m^n)$

~~Given any coherent sheaf $\mathcal{F}$ on X, $\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t.~~

~~$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $\forall\, i>0$ and $n \geq n_0$~~

2) [Asymptotic Riemann - Roch] $n = \dim X$.

$$\chi(X, \mathcal{L}^{\otimes m}) = \frac{L^n}{n!} m^n + O(m^{n-1})$$

More generally, for any coherent sheaf $\mathcal{F}$ on X,

$$\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \operatorname{rank}(\mathcal{F}) \cdot \frac{L^n}{n!} \cdot m^n + O(m^{n-1}).$$

3) [Cartan - Serre - Grothendieck Thm]

The following are equivalent.

2.a) L is ample

2.b) $\forall\, \mathcal{F}$ coherent sheaf on X, $\exists\, m_1 \in \mathbb{Z}_{>0}$ s.t.

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0, \ \forall\, i>0, \ \forall\, m \geq m_0$$

2.c) $\forall\, \mathcal{F}$ coherent sheaf on X, $\exists\, m_2 \in \mathbb{Z}_{>0}$ s.t. $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated for $\forall\, m \geq m_0$

Recall - Let $\mathcal{F}$ be a coherent sheaf. We say that $\mathcal{F}$ is globally generated if for any $x \in X$, $\Gamma(X, \mathcal{F}) \otimes \mathcal{O}_{X,x} \longrightarrow \mathcal{F}_x$ is an isom. as $\mathcal{O}_{X,x}$-module.

Fact -

Let $L \to X$ be a line bundle over a proj. variety X. Then L is ample iff $L|_{X_i}$ are ample, where X_i's are irreducible components of X.

Cor1 - (Finite pull-back)

Let $f: X \to Y$ be a finite morphism between irreducible proj. varieties. Let $L \to Y$ be a line bundle. ~~then~~ If L is ample, ~~iff~~ then f^*L is ample.

Proof. - Let $\mathcal{F}$ be a coherent sheaf on X. As f is finite and thus affine, we have $R^i f_* \mathcal{F} = 0$ for $i > 0$. Hence, by projection ~~On the other~~ formula, $\forall m \in \mathbb{Z}$, we have

$$R^i f_*(\mathcal{F} \otimes f^*L^{\otimes m}) = R^i f_* \mathcal{F} \otimes L^{\otimes m} = 0 \text{ for } i > 0.$$

In particular, by Leray's spectral sequence thm, we get

$$H^i(X, \mathcal{F} \otimes f^*L^{\otimes m}) = H^i(Y, f_*\mathcal{F} \otimes L^{\otimes m})$$
$$= 0 \text{ for } i > 0 \text{ and } m > m_0$$

Hence, f^*L is ample. $\boxtimes$

~~thm~~ Cor2: X proj. L semi-ample s.t. $L \cdot C > 0$, $\forall C$. Then L ample.

C) Nakai-Moishezon-Kleiman criterion

Thm - Let $L \to X$ be a line bundle on a projective variety X. Then L is ample iff $\forall Y \subseteq X$ ~~de~~ irreducible closed subvar. with $\dim^r \geq 1$, we have $(L|_Y)^r > 0$.

Proof. - WLOG, we may assume that X is irreducible.

"$\Rightarrow$" This follows from Cor. in A).

"$\Leftarrow$". Since every line bundle over a pt is ample, we may assume $\dim X \overset{n}{=}$ > 0 and the statement holds for $\leq n-1$.

Step 1 - $h^i(X, \mathcal{L}^{\otimes m})$ is a constant for $m \gg 0$ and $i \geq 2$.

Choose $\mathcal{L}_1$ a very ample line bundle s.t. $L \otimes L_1 = L_2$ is very ample.

Let $0 \neq s_1 \in \Gamma(X, \mathcal{L}_1)$ and $0 \neq s_2 \in \Gamma(X, \mathcal{L}_2)$ be fixed global sections.

Define $\mathcal{L}_1^{-1} \xrightarrow{\times s_1} \mathcal{J}_1 \subseteq \mathcal{O}_X$, $\mathcal{L}_2^{-1} \xrightarrow{\times s_2} \mathcal{J}_2 \subseteq \mathcal{O}_X$.

~~Let Y_1 (resp. Y_2) be the closed subvar. defined by $\mathcal{J}_1$, $\mathcal{J}_2$. then~~

~~Then we have~~

$$\cancel{0 \to \mathcal{L}^{\otimes m} \otimes \mathcal{L}_1^{-1} \xrightarrow{\times s_2} \mathcal{L}^{\otimes m+1} \longrightarrow}$$

$$\left(0 \to \mathcal{L}_2^{-1} \xrightarrow{\times s_2} \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J}_2 \to 0 \right).$$

$$\Downarrow \otimes \mathcal{L}^{\otimes m+1}$$

$$0 \to \mathcal{L}^{\otimes m} \otimes \mathcal{L}_1^{-1} \xrightarrow{\times s_2} \mathcal{L}^{\otimes m+1} \longrightarrow \mathcal{O}_X/\mathcal{J}_2 \otimes \mathcal{L}^{\otimes m+1} \to 0$$

$(*)$ $\qquad \|$

$$0 \to \mathcal{L}^{\otimes m} \otimes \mathcal{L}_1^{-1} \xrightarrow{\times s_1} \mathcal{L}^{\otimes m} \longrightarrow \mathcal{O}_X/\mathcal{J}_1 \otimes \mathcal{L}^{\otimes m} \to 0$$

$$\Uparrow \otimes \mathcal{L}^{\otimes m}$$

$$\left(0 \to \mathcal{L}_1^{-1} \xrightarrow{\times s_1} \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J}_1 \to 0 \right)$$

$Y_i \xhookrightarrow{\iota_i} X$

Let Y_i be the closed subvariety defined by $\mathcal{O}_X/\mathcal{J}_i$. Then.

$$H^j(\mathcal{O}_X, \mathcal{O}_X/\mathcal{J}_i \otimes \mathcal{L}^{\otimes m}) \underset{\text{proj. formula}}{=} H^j(Y_i, \iota_i^*(\mathcal{O}_X/\mathcal{J}_i \otimes \mathcal{L}^{\otimes m})).$$

Since $\mathcal{L}|_{Y_i}$ is ample by induction, we thus have

$$H^j(Y_i, \iota_i^*(\mathcal{O}_X/\mathcal{J}_i \otimes \mathcal{L}^{\otimes m})) \underset{\text{Serre vanishing.}}{=} 0 \text{ for } j \geq 1.\ m \gg 0$$

Hence, by the long exact sequences induced by $(*)$, for $m \gg 0$, we have.

$$H^i(X, \mathcal{L}^{\otimes m+1}) = H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{L}_1^{-1}) = H^i(X, \mathcal{L}^{\otimes m})$$

for $\forall\ i \geq 2$.

<u>step 2</u> $\underline{h^0(X, \mathcal{L}^{\otimes m}) \neq 0 \text{ for } m \gg 0}$.

By asymptotic Riemann-Roch Thm and step 1, we have

$$\chi(X, \mathcal{L}^{\otimes m}) = h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) + C$$

$$= \frac{L^{n}(>0)}{n!} m^n + O(m^{n-1}) \to +\infty \text{ as } m \to +\infty$$

for $m \gg 0$. Hence $h^0(X, \mathcal{L}^{\otimes m}) \to +\infty$ as $m \to +\infty$.

<u>step 3</u> <u>L is ~~globally generated~~ semi-ample</u>.

After replacing L by $L^{\otimes m}$ for some $m \gg 0$. We may assume that $H^0(X, \mathcal{L}) \neq 0$. Let $0 \neq s \in H^0(X, \mathcal{L})$ and let $\mathcal{J} = \mathrm{Im}(\mathcal{L}^{-1} \xrightarrow{\times s} \mathcal{O}_X)$.

Let $T \xhookrightarrow{i_T} X$ be the closed subvariety defined by $\mathcal{J}$.

Then clearly we have $\mathrm{Bs}|L| \subseteq T$. For $\forall\ m \in \mathbb{Z}$, consider

$$(**) \quad 0 \to \mathcal{L}^{\otimes m-1} \xrightarrow{\times s} \mathcal{L}^{\otimes m} \longrightarrow \mathcal{O}_X/\mathcal{J} \otimes \mathcal{L}^{\otimes m} \to 0.$$

As in step 2, for $m \gg 0$, $H^i(X, \mathcal{O}_X/\mathcal{J} \otimes \mathcal{L}^{\otimes m}) = 0$ for $\forall\ i \geq 1$.

Hence, the long exact sequence induced by $(**)$ implies

$$(***) \quad H^1(X, \mathcal{L}^{\otimes m-1}) \longrightarrow H^1(X, \mathcal{L}^{\otimes m})$$

is surjective. However, as $H^1(X, \mathcal{L}^{\otimes m})$ is finite dimensional (X proj.)

thus, $(***)$ is an isom for $m \gg 0$. In particular,

$$H^0(X, \mathcal{L}^{\otimes m}) \twoheadrightarrow H^0(X, \mathcal{O}_X/\mathcal{J} \otimes \mathcal{L}^{\otimes m})$$

for $m \gg 0$. However, as $i_T^*(\mathcal{O}_X/\mathcal{J} \otimes \mathcal{L}^{\otimes m})$ is globally generated for $m \gg 0$ by induction, for $\forall\ x \in T$, we can find $s \in H^0(X, \mathcal{L}^{\otimes m})$ s.t. $s(x) \neq 0$ and hence $\mathrm{Bs}|L^{\otimes m}| = \phi$. $m \gg 0$

<u>step 4 - Conclusion</u>.

By Cor 2 in B1. ⊠

Cor ~~2.6.?~~ - (finite pull-back)

Let $f: X \longrightarrow Y$ be a finite morphism between irreducible projective varieties. $L \to Y$ a line bundle. Then L is ample iff f^*L is ample.

Remark ~~2.6.?~~

1) Nakai - Moishezon - Kleiman's Thm can be easily generalised to Cartier $\mathbb{Q}$-divisors on irreducible normal projective varieties. Indeed, given D a Cartier $\mathbb{Q}$-divisor, then we can find $m \in \mathbb{Z}_{>0}$ divisible enough st mD is Cartier.

2) However, the Cartier $\mathbb{R}$-divisor, we have more works to do.

3) Since the normalisation is finite, it is harmless to restrict ourselves to normal varieties.

D) Ample cone

Recall - A Cartier $\mathbb{F}$-divisor D is ample if $D \sim_{\mathbb{F}} \sum a_i A_i$, with $a_i \in \mathbb{F}_{>0}$ and A_i ample Cartier divisors.

~~Prop.-~~

~~1) (Nakai's inequality)~~

~~Let D be a Cartier any $\mathbb{F}$-divisor~~

Prop - Let X be an irreducible normal projective variety.

1) (Nakai's inequality)

Let D be an ample Cartier $\mathbb{F}$-divisor on X. Then $\exists$ $\varepsilon > 0$ s.t.

$$D^{\dim(Y)} \cdot Y \geq \varepsilon$$

for any irreducible closed subvar. $Y \subseteq X$.

2) The amplitude of a Cartier $\mathbb{F}$-divisor depends only on its numerical equiv. class.

3) (Openess of amplitude)

Let D be an ample Cartier $\mathbb{F} = \mathbb{R}$ divisor. Then for any finite Cartier $\mathbb{F}$ divisors E_i, the divisor $D + \sum \varepsilon_i E_i$ is ample for $0 \leq |\varepsilon_i| << 1$.

Proof.-

1) $D \sim \Sigma a_i A_i$, $a_i > 0$. Then we have

$$D^{\dim(Y)} \cdot Y = (\Sigma a_i A_i)^{\dim Y} \geq (\Sigma a_i)^{\dim Y}$$

2) We need to show that if A, B ~~is an ample~~ are $\mathbb{F}$-Cartier $\mathbb{F}$-divisor, ~~B a Cartier~~ ~~$\mathbb{F}$-d~~ s.t. A is ample and $B \equiv 0$, then $A + B$ is ample.

This is true for $\mathbb{F} = \mathbb{Z}$ or $\mathbb{Q}$ by Nakai-Moishezon-Kleiman thm.

Now we assume that $\mathbb{F} = \mathbb{R}$ and $B = \Sigma r_i B_i$ with $r_i \in \mathbb{R}$, $B_i \in \mathrm{Div}_{\mathbb{Q}}(X)$

Let $C_1, \cdots, C_r$ be integral curves s.t. $\{C_1, \cdots, C_r\}$ form a basis of $N_1(X)_{\mathbb{R}}$. Then $B \equiv 0$ means.

$$\begin{cases} \sum_i r_i \underbrace{(B_i \cdot C_1)}_{\in \mathbb{Z}} = 0. \\ \quad \vdots \\ \sum_i r_i \underbrace{(B_i \cdot C_r)}_{\in \mathbb{Z}} = 0 \end{cases} \qquad (*)$$

$\Rightarrow \vec{r} = (r_i)$ is a solution of $\mathbb{Z}$-linear equations. Thus $\vec{r}$ is a $\mathbb{R}$-linear combination of $\mathbb{Z}$-solutions of $(*)$

$\Rightarrow$ We can write $B = \Sigma r_i' B_i'$ s.t. $r_i' \in \mathbb{R}$, $B_i' \in \mathrm{Div}(X)$ and $B_i' \equiv 0$

~~Hence, we may assume that both A and B~~

~~and $A + B = \Sigma a_i A_i + \Sigma r_j' B_j'$.~~

~~$= \Sigma (a_j' \cdot$~~

Thus we only need to prove that if A and $B \in \mathrm{Div}(X)$, s.t. A ample and $B \equiv 0$, then $A + rB$ ample for $\forall r \in \mathbb{R}$.

Fix rational numbers $r_1 < r < r_2$ and $t \in [0, 1)$ s.t. $r = t r_1 + (1-t) r_2$.

$$\text{Then} \quad A + rB = t \underbrace{(A + r_1 B)}_{\in \mathrm{Div}_{\mathbb{Q}}(X)} + (1-t) \underbrace{(A + r_2 B)}_{\in \mathrm{Div}_{\mathbb{Q}}(X)}$$

and we are done!

3) ~~$\mathbb{F} = \mathbb{Z}$ or $\mathbb{Q}$. OK because mA~~

3) WLOG, we may assume that $E_i \in \mathrm{Div}(X)$, otherwise we write ~~it~~ them as a $\mathbb{R}$-linear combination of Cartier divisors ~~and replace~~

Then write $D = \sum a_i D_i$ with $D_i \in \mathrm{Div}(X)$ and $a_i \in \mathbb{R}_{>0}$. choose $0 < c < a_1$, $c \in \mathbb{Q}$

then we have $D + \sum \varepsilon_i E_i = (cD_1 + \sum \varepsilon_i E_i) + (a_1 - c) D + \sum_{i \geq 2} a_i D_i$

Thus we may assume $D \in \mathrm{Div}(X)$. Then $\exists\, m_0 \geq 0$ s.t. $mD \pm E_i$ ample for any $m \geq m_0$. i.e. $D \pm \frac{1}{m_0} E_i$ are amples. For $0 \leq |\varepsilon_i| \ll 1$, we can write $D + \sum \varepsilon_i E_i = \mathbb{R}$-linear combination of D and $D \pm \frac{1}{m_0} E_i$ ∎

Def. - Let X be an irreducible normal projective variety. The ample cone $\mathrm{Amp}(X)$ of X is ~~closed~~ defined to be the convex cone ~~gen~~ in $N^1(X)_{\mathbb{R}}$ generated by ample $\mathbb{Q}$-Cartier divisors.

3) $\Rightarrow$ $\mathrm{Amp}(X)$ is open.

§3. Nefness and Kleiman's Thm

A) Nefness

Def.-

1) Let $L \to X$ be a line bundle on ~~an~~ proj. variety X. We say that L is <u>nef</u> if $L \cdot C \geq 0$ for $\forall\, C \subseteq X$ irreducible proj. curve.

2) Let D be a Cartier $\mathbb{F}$-divisor on an irred. proj. normal variety X. We say that D is <u>nef</u> if $D \cdot C \geq 0$ for $\forall\, C \subseteq X$ irred. proj. curve.

Basic facts

1) Nefness is preserved under pull-back.

2) If $X \to Y$ is surjective, the $L \to Y$ is ~~ample~~ nef $\iff f^*L$ is nef.

B) Kleiman's Thm.

Thm - (Kleiman)

Let X be an irreducible normal projective variety of dim n, D a Cartier $\mathbb{F}$-divisor on X. If D is nef, then $D^{\dim(Y)} \cdot Y \geq 0$ for $\forall\, Y \subseteq X$ irred. closed subvariety.

Proof.

$n = 1$, it is trivial and we thus assume that it holds $\leq n-1$. $\overset{\text{want}}{\Rightarrow} D^n \geq 0$

<u>Case 1 - $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, i.e. $\mathbb{F} = \mathbb{Q}$.</u>

Fix an ample divisor A on X, for $t \in \mathbb{R}$, consider

$$P(t) := (D + tA)^n \quad \in \mathbb{R}[t]$$
$$= \sum_{k=0}^{n} \binom{n}{k} D^k A^{n-k}\, t^{n-k}.$$

<u>Claim 1. $D^k \cdot A^{n-k} \geq 0$ for $k \leq n-1$.</u>

Indeed, we may assume A is ~~very ample~~ nonsingular and $\mathrm{Bs}|A| = \phi$. Using Bertini's Thm and the restriction,

$$D^k \cdot A^{n-k} = (D|_Y)^k \geq 0 \text{ by induction}$$

where Y is a nonsing subvar. of dimension $k \leq n-1$

Assume to the contrary that $P(0) = D^n < 0$. Then Claim 1 above implies that there exists exactly one $t_0 > 0$ s.t. $P(t_0) = 0$

<u>Claim 2 For $\forall t \in \mathbb{Q}$, $t > t_0$, $D + tA$ is ample</u>

- Let $Y \subseteq X$ be an irreducible closed subvar. of dimension $k \le n-1$, then

$$(D+tA)^k \cdot Y = \sum_{i=0}^{k} t^{k-i} D^i A^{k-i} \cdot Y$$

$$= \sum_{i=0}^{k} t^{k-i} \left(D^i|_Y \cdot (A|_Y)^{k-i} \right) \begin{cases} \geq 0 & \text{as in claim 1 + induction if } i > 0 \\ > 0 & i = 0 \end{cases}$$

$$\underbrace{}_{\cancel{\geq 0 \text{ as in claim 1 + induction}}}$$

$$> 0$$

- $Y = X$, then $P(t) > 0$ as $t > t_0$.

$\overset{\text{NMK Thm}}{\Longrightarrow}$ $D + tA$ is ample.

<u>Claim 3 $P(t_0) > 0$ and hence a contradiction.</u>

Let $Q(t) = D \cdot (D+tA)^{n-1}$, $\quad R(t) = tA \cdot (D+tA)^{n-1}$.

Then $P(t) = Q(t) + R(t)$.

- For $t > t_0$ and $t \in \mathbb{Q}$, $D + tA$ is ample $\Rightarrow D \cdot (D+tA)^{n-1} \geq 0$ as in Claim 1

Hence $Q(t_0) \geq 0$ by continuity.

- $R(t) = \sum_{k=1}^{n} \binom{n-1}{k-1} \underbrace{D^{k-1} \cdot A^{n-k+1}}_{\geq 0} t^k$ and $\underbrace{A^n}_{\text{coeff. of } t^n} > 0$.

$\Rightarrow R(t_0) > 0$ as $t_0 > 0$. $\Rightarrow P(t_0) = Q(t_0) + R(t_0) > 0$.

<u>Case 2 - $D \in \mathrm{Div}_{\mathbb{R}}(X)$, i.e. $\mathbb{F} = \mathbb{R}$.</u>

As $\mathrm{Amp}(X) \subseteq N^1(X)_{\mathbb{R}}$ is an open cone. Thus $[D] + \mathrm{Amp}(X)^{\setminus\{0\}} \subseteq N^1(X)_{\mathbb{R}}$ is open subset of $N^1(X)_{\mathbb{R}}$. Hence, $\exists \underbrace{H_1, \cdots, H_r}_{\text{ample}} \in \mathrm{Div}(X)$ s.t. $\{[H_1], \cdots, [H_r]\}$ form a basis of $N^1(X)_{\mathbb{R}}$ and $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_r) > 0$ s.t.

$$[D] + \sum \varepsilon_i [H_i] = [D(\varepsilon_1, \cdots, \varepsilon_n)] \in \mathrm{Div}_{\mathbb{Q}}(X).$$

Moreover, we can choose $\varepsilon_i < \frac{1}{n}$ for any given $n \in \mathbb{Z}_{>0}$. Moreover,

$$D(\varepsilon_1, \cdots, \varepsilon_n)^k \cdot Y > 0 \text{ for any } Y \subseteq X \text{ irred. closed subvar.}$$

Hence $D^k \cdot Y \geq 0$ by letting $\varepsilon_i \to 0$.

c) Nefness = limits of ampleness.

Prop.-

Let X be an irreducible normal proj. variety. ~~D a nef Cartier $\mathbb{R}$-divisor~~. A an ample Cartier $\mathbb{R}$-divisor. Then $D+\varepsilon A$ is ample for any $0<\varepsilon$.

$$\Updownarrow$$

D is nef.

Proof.

"$\Rightarrow$" trivial as $D\cdot C = \lim_{\varepsilon\to 0} \underbrace{(D+\varepsilon A)\cdot C}_{>0} \geq 0$

"$\Leftarrow$" • Firstly, we assume that $D \in \mathrm{Div}_{\mathbb{Q}}(X)$. Then choose $0<r<\varepsilon$ with $r \in \mathbb{Q}$.

~~then $D+\varepsilon A = D+rA+(\varepsilon-r)A$~~

Write $A = \sum r_i A_i$ with $A_i \in \mathrm{Div}(X)$ ample and $r_i \in \mathbb{R}$.

choose $0<a_i<\varepsilon r_i$ with $a_i \in \mathbb{Q}$, then $D+\varepsilon A = D + \sum a_i A_i + \sum_i (\varepsilon r_i - a_i)A_i$

Apply NMK thm to $D+\sum a_i A_i$, we know that $D+\sum a_i A_i$ is ample and hence $D+\varepsilon A$ is ample.

• Now we assume that $D \in \mathrm{Div}_{\mathbb{R}}(X)$.

Let $H_1, \dots, H_r \in \mathrm{Div}(X)$ be ample divisors s.t. $[H_i]$'s form a basis of $N^1(X)_{\mathbb{R}}$. For any given $m \in \mathbb{Z}_{>0}$, choose $\varepsilon_i(m)$ s.t. $0<\varepsilon_i(m) < \frac{1}{m}$ and

$$D + \sum_{i=1}^{r} \varepsilon_i(m) H_i = D(\varepsilon_i(m)) \in \mathrm{Div}_{\mathbb{Q}}(X) \text{ and ample}$$

Given $\forall\, \varepsilon \in \mathbb{R}_{>0}$, as $\mathrm{Amp}(X)$ is open, $\exists\, m \in \mathbb{Z}_{>0}$ s.t.

$$\left[\varepsilon A - \sum_{i=1}^{r} \varepsilon_i H_i\right] \in \mathrm{Amp}(X)$$

for any $0<\varepsilon_i<\frac{1}{m}$. Hence,

$$D+\varepsilon A = \underbrace{D + \sum_{i=1}^{r} \varepsilon_i(m) H_i}_{\text{ample}} + \underbrace{\varepsilon A - \sum_{i=1}^{r} \varepsilon_i H_i}_{\text{ample.}}$$

Hence, $D+\varepsilon A$ is ample.

DI Intersection of Cartier divisors with subvarieties

Let $Y \subseteq X$ be an irred closed subvariety in a normal proj. variety.

Let D be a Cartier divisor on X s.t. $D = (\cup U_i, f_i)$.

Then $D|_Y$ or $D \cap Y$ or $D \cdot Y$ can be understood as:

1) $L_D|_Y$ as restriction of line bundle.

2) If $Y \nsubseteq Supp(D)$, then $D|_Y = (\cup (U_i \cap Y, f_i|_{U_i \cap Y})) = D \cap Y$.

   In particular, if Y is normal, then $D|_Y$ is a Cartier div.

3) If $Y \subseteq Supp(D)$, we can choose very ample H_1, H_2

   s.t. $D = H_1 - H_2$ and $Y \nsubseteq Supp(H_1) \cup Supp(H_2)$

   Then $D|_Y = H_1|_Y - H_2|_Y$.

Note $L_D|_Y = L_{(D|_Y)} = L_{(H_1|_Y)} \otimes L^{-1}_{(H_2|_Y)}$

In particular, if X is nonsingular, $Bs|V| = \phi$, $|V|$ a linear system.

by Bertini's Thm, $H_1 \in |V|$ general $\Rightarrow$ H_1 is nonsing

as the subscheme defined by $\mathcal{O}_X(-H_1) = I_{H_1} \subseteq \mathcal{O}_X$.

We can continue, $Bs(|V||_{H_1})$ ~~is again~~ $= \phi$, then $\exists\ \hat{H}_2 \in |V||_{H_1}$

s.t. $\hat{H}_2$ is nonsingular. As $|V| \longrightarrow |V||_{H_1}$ is surjective, $\exists$

H_2 s.t. $\hat{H}_2 = H_2|_{H_1} = H_2 \cap H_1$ nonsingular. Moreover, as

$\hat{H}_2$ is general, we may ~~assume~~ choose H_2 so that H_2 is also nonsingular.

continue.

$\rightsquigarrow$ $H_1, H_1 \cap H_2, H_1 \cap H_2 \cap H_3, \ldots$

and $H^n = \left(H\big|_{H_1}\right)^{n-1} = \left(H\big|_{H_1\cap H_2}\right)^{n-2} = \left(H\big|_{H_1\cap H_2\cap H_3}\right)^{n-3}$

<u>Example</u> - (Mumford, Subramanian).

Let C be a nonsing. proj. curve of genus $g \geq 2$. Then for $\forall\ r \geq 1$, $\exists\ X \xrightarrow{\pi} C$ a fibration ~~and~~. a line bundle $L \to X$

1) $\pi^{-1}(c) \cong \mathbb{P}^r$, $\forall\ c \in C$

2) $L\big|_{\pi^{-1}(c)} = \mathcal{O}_{\mathbb{P}^r}(1)$

3) $H^0(X, L^{\otimes m}) = 0$, $\forall\ m \geq 1$

4) $H\big|_Y$ is ample, $\forall\ Y \subsetneq X$ irred proper closed ~~proj.~~ subvar.

$\Rightarrow H^{\dim X} = 0$ and H is not ample !

§4. Ample cone and Nef cone.

A) Definition

Def. - Let X be an irreducible normal projective variety.

1) $\mathrm{Amp}(X) \subseteq N^1(X)_{\mathbb{R}}$ = convex cone generated by ample Cartier $\mathbb{R}$-divisors.
2) $\mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$ = convex cone generated by nef Cartier $\mathbb{R}$-divisors.

Basic facts

1) $\mathrm{Amp}(X) \subseteq \mathrm{Nef}(X)$
2) $\mathrm{Amp}(X)$ is open and $\mathrm{Nef}(X)$ is closed.

B) Reformulation of Kleiman's Thm.

Thm - (Variant of Kleiman's Thm).

1) $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$.
2) $\mathrm{Amp}(X) = \underset{\uparrow \text{ interior}}{\mathrm{int}}(\mathrm{Nef}(X))$.

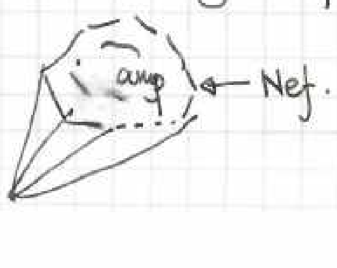
Proof.

Clearly $\overline{\mathrm{Amp}(X)} \subseteq \mathrm{Nef}(X)$ and $\mathrm{Amp}(X) \subseteq \mathrm{int}(\mathrm{Nef}(X))$.

- On the other hand, for any $[D] \in \mathrm{Nef}(X)$, $[A] \in \mathrm{Amp}(X)$, we have $D + \varepsilon A$ for $\forall\, \varepsilon > 0 \Rightarrow [D] \in \overline{\mathrm{Amp}(X)}$.
- $[D] \in \mathrm{int}(\mathrm{Nef}(X))$, $[A] \in \mathrm{Amp}(X) \Rightarrow \exists\, \varepsilon > 0$ s.t. $D - \varepsilon A$ nef. thus $D = \underbrace{D - \varepsilon A}_{\text{nef}} + \underbrace{\varepsilon A}_{\text{ample}}$ is ample.

Remark -

In general, if $\rho(X) \geq 3$, the cones are very complicated!!
e.g. not polyhedral in general.



c) Kleiman-Mori cone.

Let X be an irreducible normal proj. variety.

Def.-

1) (Cone of divisors)

$NE(X) \subseteq N_1(X)_{\mathbb{R}}$ = convex cone generated by irred proj. curves.

2) (Kleiman-Mori cone)

$\overline{NE}(X)$ = closure of $NE(X)$ is called the <u>Kleiman-Mori cone of X</u>

Remark - ⚠

~~NE(X)~~ In general $NE(X)$ is neither closed nor open.

Thm - (Duality via cones).

Let X be an irreducible normal proj. var. X.

1) $\delta \in N^1(X)_{\mathbb{R}}$ is nef $\Leftrightarrow$ $\delta|_{\overline{NE}(X)} \geq 0$ i.e. $\mathrm{Nef}(X) \underset{\text{dual cone}}{\longleftrightarrow} \overline{NE}(X)$

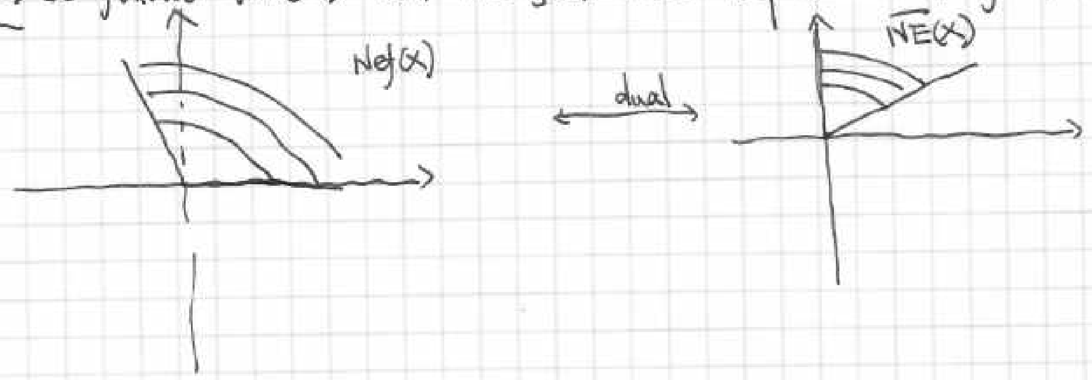
2) $\delta \in N^1(X)_{\mathbb{R}}$ is ample $\Leftrightarrow$ $\delta|_{\overline{NE}(X)\setminus\{0\}} > 0$.

Proof.-

1) Let $\alpha \in \overline{NE}(X)$, Then $\exists\, \alpha_i \in NE(X) \to \alpha$. $\Rightarrow \delta\cdot\alpha = \lim_i \delta\cdot\alpha_i \geq 0$.

~~2) $\delta \in N^1(X)_{\mathbb{R}}$ is ample $\Leftrightarrow$ $\exists\, \delta'$ ample, $\exists\, \varepsilon > 0$ st $\delta - \varepsilon\delta'$ nef $\Leftrightarrow$ $\delta - \varepsilon\delta'|_{\overline{NE}(X)} \geq 0$~~

2) It follows from 1) and the fact that $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$.



~~Cor. Let D be an ample Cartier divisor on~~

Remark

In general, $\delta|_{NE(X)\setminus\{0\}} > 0 \not\Rightarrow \delta$ is ample if $NE(X)$ is not closed

~~Fact $\delta\cdot C \geq \varepsilon$, for for $\varepsilon > 0$, then δ ample.~~

Chapter III Bigness and pseudoeffectivity.

§1 Iitaka dimension

A) Definition

Let X be an irreducible ~~normal~~ projective variety, $L \to X$ a line bundle.

For any $m \in \mathbb{Z}_{>0}$, consider

$$\Phi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

Let Y_m to be the closure of the image of X.

<u>Def.</u> - (Iitaka dimension)

The <u>Iitaka dimension</u> ~~κ~~ $\kappa(X,L)$ of L is defined as $\begin{cases} \max\limits_{m \in \mathbb{Z}_{>0}} \dim Y_m \\ \text{or} \\ -\infty \text{ if } H^0(X, L^{\otimes m}) = 0 \text{ for any } m \in \mathbb{Z}_{>0} \end{cases}$

~~X normal~~

<u>Remark</u> - (Kodaira dimension)

The <u>kodaira dimension</u> $\kappa(X)$ of an irred. proj. variety is defined to be the Iitaka dimension $\kappa(\hat{X}, K_{\hat{X}})$ of any resolution $\hat{X} \to X$, where $K_{\hat{X}}$ is the canonical bundle of $\hat{X}$, i.e. $K_{\hat{X}} = \Lambda^{\dim(\hat{X})} \Omega^1_{\hat{X}}$.

This is the most basic birational invariant of X.

<u>Example</u>

If L is semi-ample, then for any $m \in \mathbb{Z}_{>0}$ s.t. $Bs|L^{\otimes m}| = \emptyset$. We have

$$\kappa(X, L) = \dim \Phi_{|L^{\otimes m}|}(X).$$

B) Iitaka fibration

<u>Thm</u> -

Let X be an irreducible normal proj. variety, $L \to X$ line bundle on X s.t. $\kappa(X, L) > 0$. ~~Then for any $m \gg 0$ s.t. $H^0(X, L^{\otimes m}) \neq 0$,~~

then $\exists$

$$\begin{tikzcd} X \arrow[d, dashed, "\phi_m"] \arrow[dr, phantom, "\circlearrowright"] & X_\infty \arrow[l, "u_\infty"'] \arrow[d, "\phi_\infty"] \\ Y_m & Y_\infty \arrow[l, dashed, "\nu_k"] \end{tikzcd}$$

for $\forall\, m \gg 0$ s.t. $H^0(X, L^{\otimes m}) \neq 0$

where 1) X_∞ and Y_∞ are ~~normal~~ irreducible normal proj. varieties, indep. of m

2) ϕ_∞ is a fibraration, i.e. surjective with connected fibres.

3) u_∞ is a birational morphism

4) η_∞ is a birational map

5) $\phi_m := \Phi_{|L^{\otimes m}|}$

6) $\dim Y_\infty = \kappa(X, L)$

7) Let F be a very general fibre of ϕ_∞, then $\kappa(F, L_\infty|_F) = 0$,
where $L_\infty = u_\infty^* L$

<u>Remark</u> -

1) "very general" means outside "countable union of proper closed subvar."

2) If L is semi-ample and $\mathrm{Bs}|L^{\otimes m}| = \phi$, then $X_\infty = X$ and (we may take)

$$X \to Y_\infty \to Y_m, \quad \phi_m = \Phi_{|L^{\otimes m}|}$$ is exactly the Stein factorisation.

<u>Def.</u> -

In the setting of the thm, $\phi_\infty : X_\infty \to Y_\infty$ is called the <u>Iitaka fibration</u> associated to L, which is unique up to birational equivalence.

<u>Cor</u> - Let L be ~~an~~ a line bundle on an irreducible normal proj. variety X and set $\kappa = \kappa(X, L)$. Then $\exists$ constants a, $A > 0$ s.t.

$$a \cdot m^\kappa < h^0(X, L^{\otimes m}) < A \cdot m^\kappa$$

for $\forall\, m \gg 0$ s.t. $H^0(X, L^{\otimes m}) \neq 0$.

<u>Remark</u> -

1) It is possible that $\kappa(X, L) > 0$ but $H^0(X, L^{\otimes m_k}) = 0$ for $m_k \to +\infty$.

2) X normal, Y normal, $Y \xrightarrow{f} X$ birational, then $\kappa(Y, f^*L) = \kappa(X, L)$.

<u>Example</u> ($\kappa(X, L)$ is not invariant under normalisation)!

$X \subseteq \mathbb{P}^2$ (plane) nodal cubic curve. Then $\exists$ $L \in \mathrm{Pic}(X)$ with $\deg L = 0$ and $H^0(X, L^{\otimes m}) = 0$ for $\forall\, m \geq 1$. However, $\hat{X} = \mathbb{P}^1 \xrightarrow[\text{normalisation}]{n} X$ with $n^* L \cong \mathcal{O}_{\mathbb{P}^1}$
i.e. "$\kappa(X, L)$" $= -\infty$ and $\kappa(\hat{X}, f^*L) = 0$

§2. Big line bundles and divisors

A) Definition

Def. — 1) Let $L \to X$ be a line bundle over an irreducible proj variety X.

Then L is called big if $\chi(X, L) = \dim X$.

2) Let D be a Cartier divisor on an irreducible normal proj. variety X.

Then D is called big if $\chi(X, L_D) = \dim X$.

Remark —

1) A priori ~~this definition does not~~ c it is NOT clear if this ~~definition~~ coincides with our previous one.

2) Iitaka dimension is NOT invariant under normalisation. However, bigness is invariant under normalisation!

Lemma — (First equivalent definition).

Let $L \to X$ be a line bundle over an irreducible projective variety.
Then L is big iff $\exists\, C > 0$ constant s.t. $h^0(X, L^{\otimes m}) \geq Cm^n$ for $\forall\, m \gg 0$ and $h^0(X, L^{\otimes m}) \neq 0$, where $n = \dim X$.

Proof. —

① If X is normal, this follows from Cor. in Chap II, §1.B.

② ~~If~~ X is not normal. ~~then~~ we taken $\nu : \hat{X} \to X$ the normalisation.
Then we have

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{\hat{X}} \to Q \to 0.$$

where Q is a cohert sheaf supported on ~~the~~ X_{sing} (~~codi~~ $\dim \leq n-1$).
Since ν is finite, by projection formula and Leray's ~~seq~~ spectral sequence, we have

$$h^0(\hat{X}, \nu^* L^{\otimes m}) = h^0(X, L^{\otimes m} \otimes \nu_* \mathcal{O}_{\hat{X}})$$

$$\leq h^0(X, L^{\otimes m}) + h^0(X, L^{\otimes m} \otimes Q).$$

$$0 \to \mathcal{O}_X \otimes L^{\otimes m} \to \nu_* \mathcal{O}_{\hat{X}} \otimes L^{\otimes m} \to Q \otimes L^{\otimes m} \to 0$$

Fact.

Let $\mathcal{F}$ be a coherent sheaf over ~~an~~ proj. variety Y of dim n. $\mathcal{L}$ a line bundle.

Then $h^i(Y, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = O(m^n)$ for $\forall i$

Hence, $h^0(\hat{X}, f^*L^{\otimes m}) \leq O(m^n) + O(m^{n-1})$

$$\overset{\text{automatic} \to}{\vee\!|}$$

$$h^0(X, L^{\otimes m}) \quad \square$$

because $h^0(X, \mathcal{Q} \otimes L^{\otimes m}) = h^0(X_{sing}, \mathcal{Q} \otimes L^{\otimes m}|_{X_{sing}})$

dim $\leq n-1$

B) Kodaira's lemma

Prop. - (Kodaira's lemma)

Let D be a Cartier divisor on an irred. normal X. $F \geq 0$ an effective Cartier divisor on X.

Then $H^0(X, \mathcal{O}_X(mD - F)) \neq 0$ for $\forall m >> 0$ s.t. $H^0(X, \mathcal{O}_X(mD)) \neq 0$

Proof. -

Let $\mathcal{I}_F = \mathcal{O}_X(-F) \subseteq \mathcal{O}_X$ be the ideal sheaf defined by F. i.e. $\mathcal{I}_F|_{U_i} = \underbrace{f_i \mathcal{O}_X}_{\text{locally}}$

~~$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X$~~

$$0 \to \mathcal{I}_F = \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_F = \mathcal{Q} \to 0$$

$\otimes \mathcal{O}_X(mD)$

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{Q} \otimes \mathcal{O}_X(mD) \to 0.$$

Then $h^0(X, \mathcal{O}_X(mD)) \geq C m^n$ for $m >> 0$ and $h^0(X, \mathcal{O}_X(mD)) \neq 0$

$$h^0(X, \mathcal{Q} \otimes \mathcal{O}_X(mD)) = O(m^{n-1}) \quad \text{as } \mathcal{Q} \text{ is supported on Supp}(F)$$

$\Rightarrow h^0(X, \mathcal{O}_X(mD)) > h^0(X, \mathcal{Q} \otimes \mathcal{O}_X(mD))$ for $m >> 0$ and $h^0(X, \mathcal{O}_X(mD)) \neq 0$.

$\Rightarrow h^0(X, \mathcal{O}_X(mD - F)) \neq 0$.

Cor. - (Second equivalent definition)

Let D be an ~~irreducible~~ Cartier divisor on an irreducible normal proj. variety X.

Then the following are equivalent:

(1) D is big

(2) For ***any*** ample divisor A, $\exists$ $m>0$ and $N\geq 0$ Weil divisor s.t.

$$mD \sim A + N$$

(3) For ***some*** ample divisor A, - - - - - - - - - - -

(4) $\exists$ an ample divisor A, $\exists m>0$ and $N\geq 0$ s.t.

$$mD \equiv A + N.$$

Proof.-

1) $\Rightarrow$ 2) Choose $r \gg 0$ s.t. $|rA| \ni H_r$ and $|(r+1)A| \ni H_{r+1}$.

Apply Kodaira's lemma to D and H_{r+1}, we get $N_{r+1}\geq 0$, $m>0$ s.t.

$$mD - H_{r+1} \sim N_{r+1} \Rightarrow mD \sim H_{r+1} + N_{r+1}$$

$$\sim A + \underbrace{H_r + N_{r+1}}_{N}.$$

2) $\Rightarrow$ 3) trivial

3) $\Rightarrow$ 4) trivial

4) $\Rightarrow$ 1) $A' = mD - N \equiv A$ and hence ample. ~~Moreover, as $N\geq 0$, we have~~

Choose $r \gg 0$ s.t. rA' is very ample, then $\Phi_{|rA'|}$ is an embedding and $\kappa(X, A') = \dim X$. On the other hand, consider

$|V| = |rA'| + rN \subseteq |mrD|$ sublinear system,

then $\dim \Phi_{|mrD|} \geq \dim \Phi_{|V|} = \dim \Phi_{|rA'|} = \dim(X)$.

Hence, D is big.

Remark

By 4), bigness is a numerical invariant property.

~~C) Bigness of nef divisors~~

C) Bigness of nef divisors

Def.- A Cartier $\mathbb{F}$-divisor $D \in \mathrm{Div}_{\mathbb{F}}(X)$ is big if it can be written in the form $D = \sum a_i D_i$, where $a_i \in \mathbb{F}_{>0}$ and $D_i \in \mathrm{Div}(X)$ big Cartier divisors.

Thm - (Siu)

Let D, E be two nef Cartier $\mathbb{Q}$-divisor on an irreducible normal projective variety X. Assume that $D^n > n\cdot D^{n-1}\cdot E$. Then $D-E$ is big.

WLOG, X is nonsingular.

Proof. - Let A be a fixed ample Cartier divisor on X. Choose $\varepsilon \in \mathbb{Q}_{>0}$ s.t.

$$\underbrace{(D+\varepsilon A)^n}_{=D'} \underset{\text{by continuity}}{>} n\cdot \underbrace{(D+\varepsilon A)}_{D'}^{n-1}\cdot \underbrace{(E+\varepsilon A)}_{E'}. \qquad \text{and } D-E = D'-E'$$

Thus, we may assume that both D and E are ample.

Moreover, choose $m\in\mathbb{Z}_{>0}$ s.t. mD and mE are very ample Cartier divisors, then we again have

$$(mD)^n = m^n D^n > n\cdot (mD)^{n-1}\cdot (mE)$$

$$\text{and} \quad (mD - mE) = m(D-E)$$

Thus we may assume that both D and E are very ample Cartier divisors. Choose $E_1, \dots, E_m \in |E|$ general, for fixed $m\in\mathbb{Z}_{>0}$ ← nonsing.

$$\mathcal{I}_{E_i} = \mathcal{O}_X(-E_i)$$

Then $\quad 0 \to \mathcal{O}_X(-\sum_{i=1}^m E_i) \longrightarrow \mathcal{O}_X \longrightarrow \bigoplus_{i=1}^m \mathcal{O}_X/\mathcal{I}_{E_i} =: \mathcal{Q}_i = \mathcal{O}_{E_i}$

$\otimes\, \mathcal{O}_X(mD)$

$\Rightarrow \quad h^0(X, \mathcal{O}_X(mD - \sum_{i=1}^m E_i)) \geq h^0(X, \mathcal{O}_X(mD)) - \sum_{i=1}^m h^0(X, \mathcal{Q}_i\otimes\mathcal{O}(mD))$

$$\underline{\underline{\text{By ARR Thm}}} \quad \frac{D^n}{n!}m^n - \sum_{i=1}^m \frac{D^{n-1}\cdot E_i}{(n-1)!} m^{n-1} + \underbrace{m\,O(m^{n-2})}_{+O(m^{n-1})}$$

$$= \left(\frac{D^n}{n!} - \frac{D^{n-1}\cdot E}{(n-1)!}\right) m^n + O(m^{n-1}).$$

$$\geq C m^n \text{ if } D^n - nD^{n-1}\cdot E > 0 \quad \blacksquare$$

Cor - (Bigness of nef divisors)

Let D be a Cartier ~~Q-~~divisor on an irreducible proj. normal variety X. If D is nef, then D is big $\Leftrightarrow D^n > 0$.

Proof. -

~~"$\Rightarrow$" by ARR thm~~

~~"$\Leftarrow$"~~

"$\Rightarrow$" WLOG, we may assume that D is Cartier and $D \sim A + N$ where A is very ample and $N \geq 0$

then
$$D^n = D^{n-1}\cdot(A+N) = A\cdot D^{n-1} + \underbrace{D^{n-1}\cdot N}_{\geq 0.}$$
$$\geq A\cdot D^{n-1}$$

choose $H \in |A|$ general, then $D|_H = \underbrace{A|_H}_{\text{ample}} + \underbrace{N|_H}_{\geq 0}$ and hence big, thus $A\cdot D^{n-1} > 0$ by induction.

"$\Leftarrow$" Apply Siu's thm to $E = 0$.

Example - (**NOT** true if D is not nef).

Let $X \xrightarrow{f} \mathbb{P}^2$ be the blowing-up at a point p with exceptional div. E. Then $E^2 = -1$, let $A = \mathcal{O}_{\mathbb{P}^2}(1)$ be the tautological line bundle. then
$$\chi(X, f^*A + mE) \geq \chi(\mathbb{P}^2, A) = 2. \quad \text{for } \forall\, m \in \mathbb{Z}_{>0}.$$

However,
$$(f^*A + mE)^2 = (f^*A)^2 + 2f^*A\cdot mE + m^2E^2.$$
$$= \underset{1}{A^2} + 0 - m^2$$
$$= 1 - m^2 < 0 \quad \text{if } m \geq 2.$$

§3. Pseudo-effective cone and big cone.

Lemma. - (Bigness is numerical)

Let D and D' be two Cartier $\mathbb{R}$-divisors on an irreducible projective normal variety X.

1) If $D \equiv D'$, then D is big iff D' is big.

2) D is big $\Leftrightarrow$ $\exists$ A ample Cartier $\mathbb{R}$-divisor and $N \geq 0$ ~~Weil~~ effective Cartier $\mathbb{R}$-divisor s.t. $D \equiv A + N$.

Proof. - only need to prove 2).

"$\Rightarrow$" $\quad D = \sum \underset{\mathbb{R}_{>0}}{a_i} \underbrace{D_i}_{\text{big Cartier}} \equiv A + N$ by second equivalent definition in Chap III § 2.B

"$\Leftarrow$" Reduce to the case $D - N \sim_{\mathbb{R}} A'$ with A' ample and $A' \equiv A$.

Reduce to the case A ample Cartier divisor, N effective Cartier divisor, the $A + sN$ is big for any $s \in \mathbb{R}_{\geq 0}$.

Ineed, if $s \in \mathbb{Q}$, this is done. For $s \in \mathbb{R} \setminus \mathbb{Q}$, choose $r_1, r_2 \in \mathbb{Q}_{>0}$ s.t. $r_1 < t < r_2$ and hence $\exists$ $t \in (0,1)$ s.t. $s = tr_1 + (1-t) r_2$. Then

$$A + sN = t\underbrace{(A + r_1 N)}_{\in \mathrm{Div}_{\mathbb{Q}}(X)} + (1-t)\underbrace{(A + r_2 N)}_{\in \mathrm{Div}_{\mathbb{Q}}(X)}$$

Cor. - (Bigness is open)

Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ be a big Cartier $\mathbb{R}$-divisors. $E_i \in \mathrm{Div}_{\mathbb{R}}(X)$, $1 \leq i \leq r$. Then

$$D + \sum_{i=1}^{r} \varepsilon_i E_i \text{ is big for any } 0 < |\varepsilon_i| \ll 1$$

Proof. - $D \equiv \underbrace{A}_{\text{ample}} + \underbrace{N}_{\geq 0}$ Then $A + \sum \varepsilon_i E_i$ is ample for any $0 < |\varepsilon_i| \ll 1$

Def. - Let X be an irreducible normal proj. variety.

1) The <u>big cone</u> $Big(X) \subseteq N^1(X)_{\mathbb{R}}$ is the convex cone generated by big Cartier $\mathbb{R}$-divisors on X.

2) the ~~pseudo~~ <u>effective cone</u> $Eff(X) \subseteq N^1(X)_{\mathbb{R}}$ is the convex cone generated by effective Cartier $\mathbb{R}$-divisors on X.

3) The <u>pseudo-effective cone</u> $\overline{Eff}(X)$ is the closure of $Eff(X)$.

Remark -

1) In general, $Eff(X)$ is neither open nor closed

~~2) A Cartier $\mathbb{R}$-divisor~~ .

Thm -

$$Big(X) = Int(\overline{Eff}(X)) \quad \text{and} \quad \overline{Big(X)} = \overline{Eff}(X).$$

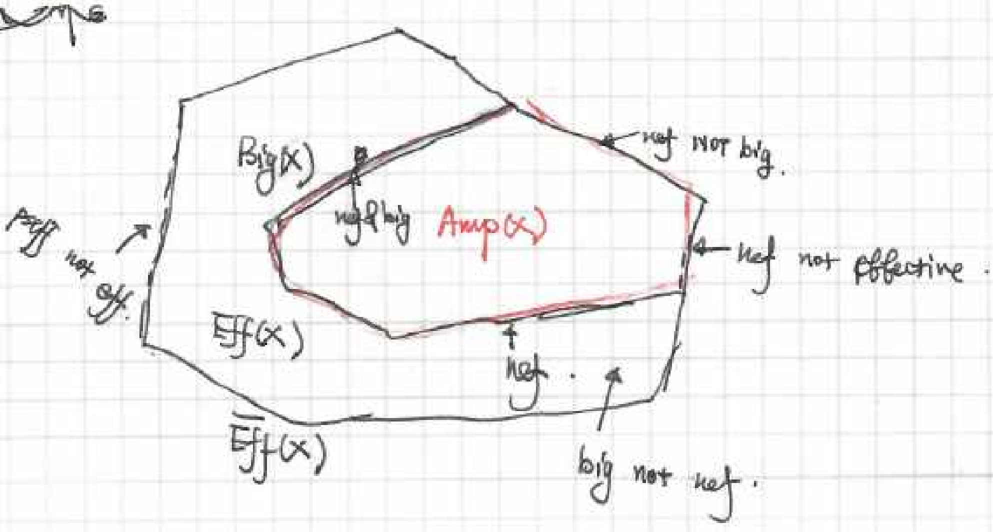
Proof

$Big(X)$ is open, $\overline{Eff}(X)$ is closed and $Big(X) \subseteq Int(\overline{Eff}(X))$

Given $\eta \in \overline{Eff}(X)$, choose $[A]$ ample class. Then.

$$\lim_{m\to+\infty} \underbrace{\eta_m + \frac{1}{m}[A]}_{\text{big}} = \eta$$

where $\eta_m \in Eff(X)$ s.t. $\lim_{m\to+\infty} \eta_m = \eta$. i.e. pseff = limit of big.



§4. Volume function

A) Volume of a line bundle

Def. - Let $L \to X$ be a line bundle over an irreducible proj. var. of dimension n. Then the volume of L is defined to be

$$\mathrm{Vol}(L) := \mathrm{Vol}_X(L) = \limsup_{m \to +\infty} \frac{h^0(X, L^{\otimes m})}{m^n / n!} \in \mathbb{R}_{\geq 0}$$

- the volume $\mathrm{vol}_X(D)$ of a Cartier divisor on an irreducible normal variety X is defined as $\mathrm{vol}_X(L_D)$.

Remark -

1) L big $\iff \limsup_{n \to +\infty} h^0(X, L^{\otimes m}) \geq C \cdot m^n \iff \mathrm{Vol}(L) > 0$.
2) If L is nef, by ARR thm, we have
$$\mathrm{Vol}(L) = L^n.$$
3) $\exists\ (X, L)$ s.t. $\mathrm{Vol}(L) \in \mathbb{R} \setminus \mathbb{Q}$. where $X = E \times E$ E = elliptic curve

B) Volume of a Cartier $\mathbb{Q}$-divisor

Lemma -

Let D be a big Cartier $\mathbb{Q}$-divisor on an irreducible normal projective variety X. Then for $\forall\, a \in \mathbb{Z}_{>0}$, we have

$$\mathrm{Vol}(aD) = a^n \mathrm{Vol}(D)$$

where $n = \dim X$.

Def. -

Let D be a Cartier $\mathbb{Q}$-divisor, i.e. $\exists\ m \in \mathbb{Z}_{>0}$ and D' Cartier divisor s.t. $mD = D'$. The volume of D is defined as.

$$\mathrm{Vol}(D) = \frac{1}{m^n} \mathrm{Vol}(D').$$

where $n = \dim X$.

C) Numerical invariance of volume.

~~Lemma -~~

~~For any Cartier divisors D~~

Lemma -

Let D be a big Cartier divisor on an irreducible normal proj. var. X of dimension n. For any Cartier divisor N on X and any $\varepsilon > 0$, $\exists$ $p_0 = p_0(N, \varepsilon)$ such that for any $p > p_0$, we have

$$\frac{1}{p^n} \left| \mathrm{Vol}(pD - N) - \mathrm{Vol}(pD) \right| < \varepsilon .$$

Prop -

If D and D' are Cartier $\mathbb{Q}$-divisors on an irreducible normal proj. var. X of dimension n s.t. $D \equiv D'$, then $\mathrm{Vol}(D) = \mathrm{Vol}(D')$.

Outline of the proof -

We only need to prove $\mathrm{Vol}(D) = \mathrm{Vol}(D+P)$, where P is a Cartier $\mathbb{Q}$-divisor s.t. $P \equiv 0$. Moreover, we may assume that both D and P are Cartier and D is big. The key point is following fact

$\exists$ N a Cartier divisor on X such that

$$H^0(X, \mathcal{O}_X(N+P)) \neq 0$$

for $\forall$ P Cartier divisor on X with $P \equiv 0$.

(1) $H^0(N - pP) \neq 0$ $\Rightarrow$ $H^0(m(p(D+P) - N)) = H^0(mpD - m(N - pP)) \xhookrightarrow{+mN'} H^0(mpD)$.

$\quad\ \Downarrow$
$\quad\ N'$

$$\Rightarrow \mathrm{Vol}(p(D+P)) \leq \mathrm{Vol}(pD) = p^n \mathrm{Vol}(D)$$

(2) ~~$H^0(N+pP) \neq 0$~~
Lemma above $\Rightarrow$ $\frac{1}{p^n} \left| \mathrm{Vol}(p(D+P) - N) - \mathrm{Vol}(p(D+P)) \right| \to 0$ as $p \to +\infty$

$$\Rightarrow \frac{1}{p^n} \mathrm{Vol}(p(D+P) - N) \to \mathrm{Vol}(D+P) \text{ as } p \to +\infty$$

$$\Rightarrow \mathrm{Vol}(D+P) \leq \mathrm{Vol}(D).$$

For the reverse inequality, replace P by $-P$, D by $D+P$. ☒

D) Continuity of Volume function

Thm -

Let X be an irreducible normal proj. variety of dimension n. Let $\|\cdot\|$ be any norm on $N^1_{\mathbb{Q}}(X)_{\mathbb{R}}$ inducing the usual topology on finite dimensional $\mathbb{R}$-vector space. Then $\exists$ a constant $C>0$ s.t.

$$|\mathrm{Vol}(\xi) - \mathrm{Vol}(\xi')| \leq C\cdot\left(\max\{\|\xi\|, \|\xi'\|\}\right)^{n-1}\cdot\|\xi-\xi'\|$$

for any $\xi, \xi' \in N^1(X)_{\mathbb{Q}}$.

Cor. -

The function $\xi \to \mathrm{Vol}(\xi)$ on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a continuous function

$$\mathrm{Vol} : N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}_{\geq 0}.$$

Remark -

1) $\mathrm{Vol}(\cdot)$ is actually $\mathcal{C}^1$ on $\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$.

2) (Increasing in effective directions).

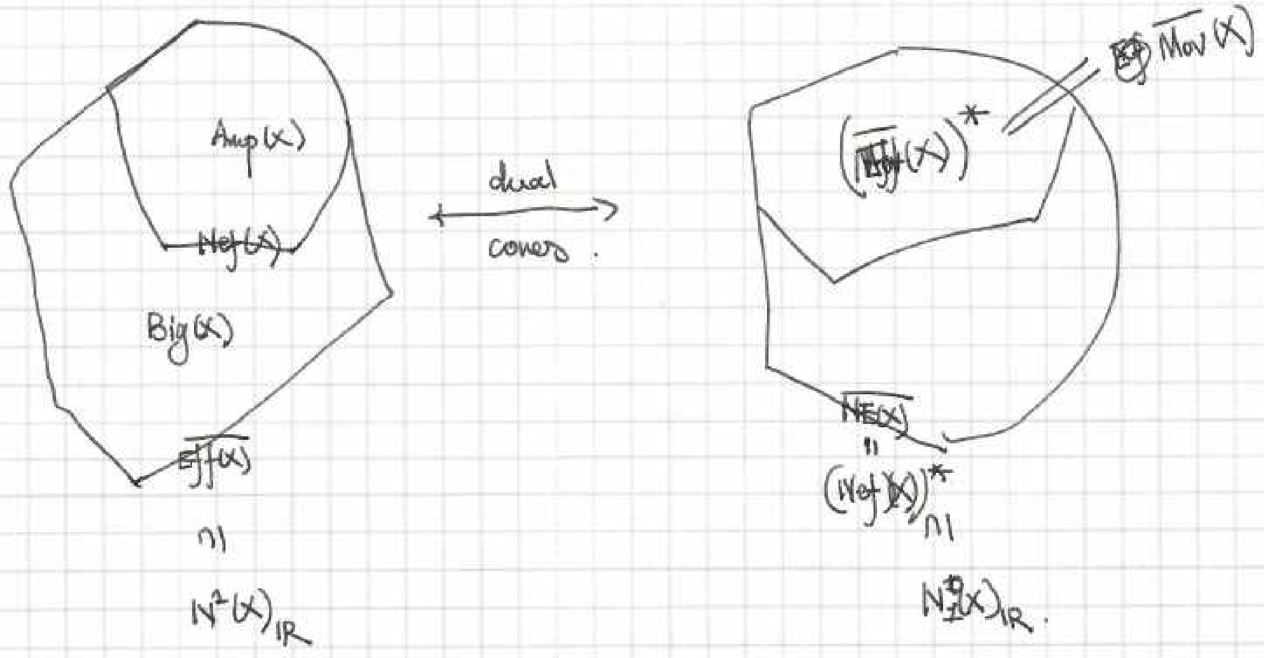
$$\mathrm{Vol}(\xi) \leq \mathrm{Vol}(\xi+e), \quad \xi \in \mathrm{Big}(X) \text{ and } e \in \overline{\mathrm{Eff}(X)}$$

3) [Birational invariance]

Let $\nu : X' \to X$ be a birational morphism between irreducible normal proj. variety. $\xi \in N^1(X)_{\mathbb{R}}$, then

$$\mathrm{Vol}(\nu^*\xi) = \mathrm{Vol}(\xi).$$

§4 The dual of the Pseudoeffective cone : BDPP



Question : What is $\overline{\mathrm{Eff}}(X)^*$?

A) mobile curves and movable cone

Def. -

1) Let X be an irreducible proj. variety of dimension n. A class $\gamma \in N_1(X)_{\mathbb{R}}$ is movable or mobile if $\exists$ a projective birational morphism $\mu : X' \to X$ and ample classes $a_1, \dots, a_{n-1} \in N^1(X')_{\mathbb{R}}$ s.t.

$$\gamma = \mu_*(a_1 \cdots\cdots a_{n-1}).$$

2) The movable cone $\overline{\mathrm{Mov}}(X) \subseteq N_1(X)_{\mathbb{R}}$ of X is the closed convex cone spanned by all movable curves.

Remark -

what does $\mu_*(a_1, \dots, a_{n-1})$ means ? choose $A_1, \dots, A_{n-1}$ very ample on X' and $D_i \in |A_i|$ s.t $\overset{C}{\shortparallel}$ $D_1 \cap \dots \cap D_{n-1}$ is a ~~curve~~ a reduced curve. Then

$$\mu_*([A_1] \cdots [A_{n-1}]) = [\mu_* C]$$

push-forward of cycles.

$\rightsquigarrow$ $\mu_*(a_1 \cdots\cdots a_{n-1})$ = extension of linear combination of very ample div. It is a well-defined map from $N_1(X')_{\mathbb{R}}$ ~~to~~ $N_1(X)_{\mathbb{R}}$ because of proj. formula.

Thm - [BDPP]

Let X be a nonsing. irred. proj. variety of dimension n. Then the cones

$$\overline{\mathrm{Mov}}(X) \quad \text{and} \quad \overline{\mathrm{Eff}}(X)$$

are dual.

Prop -. D nef divisor. Then D is big $\iff \exists\, N \geq 0$ s.t. for $\forall\, m \gg 1$, $\underbrace{D - \frac{1}{m}N}_{= A_m}$ ample

$D = A_m + \frac{1}{m} N$

B) Fujita approximation

Def. - (Fujita approximation of a big class)

Let X be an irreducible normal proj. variety of dim n. Let $\xi \in N^1(X)_{\mathbb{R}}$ be a big class. A Fujita approximation for ξ consists of

- a birational morphism $\mu : X' \to X$ with X' irred, normal, proj.
- a decomposition $\mu^* \xi = a + e$ with $\overset{\mathbb{Q}}{a} \in \mathrm{Amp}(X')$ and $e \in \overline{\mathrm{Eff}}(X)$.

Thm - (Fujita's approximation Thm)

Let X be an irreducible normal proj. variety of dimension n. $\overset{\mathbb{Q}}{\xi} \in$ ~~Amp~~ Big (X').

For any $\varepsilon > 0$, $\exists$ a Fujita approximation

$$\mu : X' \to X \quad \text{and} \quad \mu^* \xi = a + e$$

such that $\mathrm{Vol}_{X'}(a) > \mathrm{Vol}_X(\xi) - \varepsilon$

$\wedge|$

$\mathrm{Vol}_X(\xi) = \mathrm{Vol}_{X'}(\mu^* \xi)$

Remark -

Roughly speaking, the volume of a big class on X can be approximated by ample classes on birational models of X.

Thm - (Asymptotic orthogonality of Fujita approximation)

Let X be an irreducible normal proj. variety of dimension n. $\overset{0}{\#}$ $\xi \in \mathrm{Big}(X)_{\mathbb{R}}$

Fix an ample divisor $h \in \mathrm{Amp}(X)$ s.t. $h \pm \xi$ is ample. Then $\exists$ a constant $C > 0$ s.t. for any Fujita approximation

$$\mu : X' \longrightarrow X \quad \text{and} \quad \mu^* \xi = a + e$$

we have

$$(a^{n-1} \cdot e)^2_{X'} \leq C \cdot (h^n)_X \cdot (\mathrm{Vol}_X(\xi) - \mathrm{Vol}_{X'}(a))$$

$$\Rightarrow \quad a_\varepsilon^{n-1} \cdot e_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

Remark -

Roughly speaking, a may be not orthogonal to e, i.e, $a^{n-1} \cdot e$ maybe non-zero.

However, $(a^{n-1} \cdot e)^2$ can be uniformly controled from above by the difference $\mathrm{Vol}_X(\xi) - \mathrm{Vol}_{X'}(a) > 0$

In particular, we can choose a sequence of Fujita ~~deco~~ approximation

$$\mu_m : X_m \longrightarrow X \quad \text{and} \quad \mu_m^* \xi = a_m + e_m$$

s.t. $(a_m^{n-1} \cdot e_m) \longrightarrow 0$ as $m \to +\infty$. i.e. $\{a_m\}$ and $\{e_m\}$ are asymptotically orthogonal.

c) Generalised type of Hodge ~~index~~ type

Thm - (Hodge index thm)

Let D_1, D_2 be two nef divisors on a nonsingular irreducible proj. surface. Then

$$(D_1 \cdot D_2)^2 \geq D_1^2 \cdot D_2^2.$$

Remark -

This is a consequence of the classical Hodge index Thm on surface, which says

If A is an ample divisor on a nonsingular irreducible proj. surface, D a divisor s.t. $A \cdot D = 0$. Then $D^2 \leq 0$ with "=" ~~iff~~ $D \equiv 0$.

Exercise

Hodge index Thm $\Rightarrow$ Thm.

Thm - (Generalised inequality of Hodge type)

Let X be an irreducible normal proj. variety of dimension n. $\delta_1, \dots, \delta_n \in \mathrm{Nef}(X)$.

Then

$$(\delta_1 \cdots \delta_n)^n \geq (\delta_1^n) \cdots\cdots (\delta_n^n).$$

D) Proof of BDPP

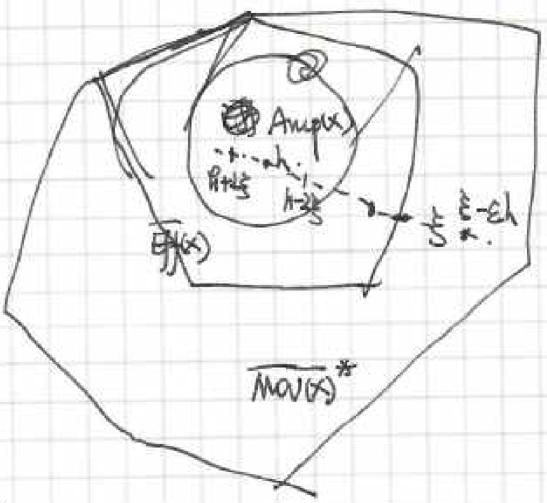
By projection formula, we have : $\mu : X' \xrightarrow{\text{bir}} X$, $a_i \in \mathrm{Amp}(X')$

$$\mu^* \xi \cdot a_1 \cdots a_{n-1} = \xi \cdot \mu_*(a_1 \cdots a_{n-1})$$

$$\geq 0 \text{ if } \xi \text{ is effective.}$$

Thus we must have $\overline{\mathrm{Eff}(X)} \subseteq \overline{\mathrm{Mov}(X)}^*$. Now we assume to the contrary that this inclusion is strict. Then $\exists\ \underset{0}{\overset{\neq}{\xi}} \in N^1(X)_{\mathbb{R}}$ s.t.

$$\xi \in \mathrm{Boundary}(\overline{\mathrm{Eff}(X)}) \text{ and } \xi \in \mathrm{interior}\left(\overline{\mathrm{Mov}(X)}^*\right).$$



choose $\underset{0}{\overset{\in \mathrm{Amp}(X)}{\underset{\neq}{h}}}$ s.t. $h \pm 2\xi$ is ample. Moreover, $\exists\ \varepsilon > 0$ s.t. $\xi - \varepsilon h \in \mathrm{Int}(\overline{\mathrm{Mov}(X)}^*)$

Thus

$$\frac{\xi \cdot \gamma}{h \cdot \gamma} \geq \varepsilon \quad \text{for any } \underset{0}{\underset{\neq}{\gamma}} \in \overline{\mathrm{Mov}(X)} \qquad (\star)$$

~~Want to show~~ $\cancel{\exists\ \gamma_\delta \in \overline{\mathrm{Mov}}(X) \text{ s.t. } \frac{\xi\cdot\gamma_\delta}{h\cdot\gamma_\delta} \to 0 \text{ as } \delta\to 0}$

~~Want to show~~ $\cancel{\exists\ 0\neq\gamma_\delta \in \overline{\mathrm{Mov}}(X) \text{ and } \xi_\delta \in \mathrm{Big}(X) \text{ s.t. } \frac{\xi\cdot\gamma_\delta}{h\cdot\gamma_\delta} \leq C\cdot \mathrm{Vol}(\xi_\delta) + \epsilon\cdot\delta \text{ and } \xi_\delta \to \xi,\ \delta \to 0}$

$\cancel{\exists\ 0\neq\xi_\delta \in \mathrm{Big}(X) \text{ s.t. } \lim_{\delta\to 0} \mathrm{Vol}(\xi_\delta) = 0 \text{ and } \xi_\delta \to \xi.}$

~~Thus~~ $\cancel{\mathrm{Vol}(\xi) = 0}$ ~~by continuity of~~ $\cancel{\mathrm{Vol}(\cdot)}$

WANT TO show $\exists\ 0 \neq \gamma_\delta \in \overline{\mathrm{Mov}}(X)$ s.t. $\frac{\xi\cdot\gamma_\delta}{h\cdot\gamma_\delta} \to 0$ as $\delta \to 0$.

Let $\xi_\delta = \xi + \delta h$, $\delta \in \mathbb{R}_{>0}$. Then ξ_δ is big. By Fujita's approximation,

$$\exists\, \mu_\delta : X_\delta \longrightarrow X \text{ and } \mu_\delta^* \xi_\delta = \underset{\mathrm{Amp}(X_\delta)}{\overset{0}{a_\delta}} + \underset{\mathrm{Eff}(X_\delta)}{e_\delta} \quad \text{s.t.}$$

$$\mathrm{Vol}_{X_\delta}(a_\delta) \geq \mathrm{Vol}_X(\xi_\delta) - \delta^{2n}$$

$$\text{and } \left(a_\delta^n\right)_{X_\delta} = \mathrm{Vol}_{X_\delta}(a_\delta) \geq \tfrac{1}{2}\mathrm{Vol}_X(\xi_\delta) \geq \tfrac{1}{2}\mathrm{Vol}_X(\delta h) = \tfrac{1}{2}\delta^n\cdot h^n$$

Set $\gamma_\delta := \mu_{\delta *}(a_\delta^{n-1})$. Then by the projection formula and generalised inequality of Hodge type, we have

$$(h\cdot\gamma_\delta)_X \underset{\text{form.}}{\overset{\text{proj}}{=}} (\mu_\delta^* h\cdot a_\delta^{n-1}) \underset{\text{Hodge}}{\geq} \Big[(\mu_\delta^* h)^n_{X_\delta}\cdot \underbrace{(a_\delta^n)_{X_\delta}\cdots(a_\delta^n)_{X_\delta}}_{n-1}\Big]^{\frac{1}{n}}$$

$$= \left(h^n\right)_X^{\frac{1}{n}} \cdot \left(a_\delta^n\right)_{X_\delta}^{\frac{n-1}{n}}$$

$$\text{and } (\xi\cdot\gamma_\delta)_X \leq (\xi_\delta\cdot\gamma_\delta)_X \underset{\text{for.}}{\overset{\text{proj.}}{=}} (\mu_\delta^*\xi_\delta \cdot a_\delta^{n-1})_{X_\delta}$$

$$= (a_\delta^n)_{X_\delta} + (e_\delta\cdot a_\delta^{n-1})_{X_\delta}$$

Now we note $h - \xi_\delta = (1-\delta)h - \xi = \cancel{(1-\delta)\left(h - \tfrac{1}{1-\delta}\xi\right)}$

is ample if $\delta < \frac{1}{2}$ as $h - 2\xi$ is ample.

$$\overset{\shortparallel}{2\left(\tfrac{1}{2}h - \xi\right)}$$

Hence, AOFA thm. in $\underline{B1}$ implies that $\exists$ a constant C_1 s.t.

$$\left(e_\delta \cdot a_\delta^{n-1}\right)_{X_\delta} \leq \Big[C_1 \cdot (h^n)_X \cdot \big(\underbrace{\mathrm{Vol}_{X_\delta}(\xi_\delta) - \mathrm{Vol}_{X_\delta}(a_\delta)}_{\leq \delta^{2n}} \big) \Big]^{\frac{1}{2}}$$

$$\leq \underbrace{C_2}_{\text{constant independent of } \delta} \cdot \delta^n$$

then
$$\frac{\xi \cdot \gamma_\delta}{h \cdot \gamma_\delta} \leq \frac{(a_\delta^n)_{X_\delta} + (e_\delta \cdot a_\delta^{n-1})_{X_\delta}}{(h^n)_X^{\frac{1}{n}} \cdot (a_\delta^n)^{\frac{n-1}{n}}}$$

$$\leq \frac{1}{(h^n)_X^{\frac{1}{n}}} \cdot (a_\delta^n)^{\frac{1}{n}} + \frac{C_2 \delta^n}{(h^n)_X^{\frac{1}{n}} \cdot \left(\frac{1}{2}\delta^n \cdot (h^n)_X\right)^{\frac{n-1}{n}}}$$

$$\leq C_3 \cdot (a_\delta^n)^{\frac{1}{n}} + C_4 \cdot \delta$$

constants independent of δ.

$\delta \to 0$, then $(a_\delta^n)^{\frac{1}{n}} = \mathrm{Vol}_{X_\delta}(a_\delta) \to \mathrm{Vol}_X(\xi) = 0$.

$\mathrm{Vol}_{X_\delta}(f_\delta^* \xi_\delta) = \mathrm{Vol}_X(\xi_\alpha)$.

Hence $\frac{\xi \cdot \gamma_\delta}{h \cdot \gamma_\delta} \to 0$ as $\delta \to 0$. A contradiction to $(\star)$ ▨

Chapter IV. Vanishing theorem and multiplier ideals

§ 1. ~~[illegible]~~ Thms. vanishing and applications

B) Kodaira's vanishing Thm

Thm - (Kodaira)

Let L be an ample line bundle over a nonsing. ~~irred.~~ proj. variety X of dimension n. Then

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0, \quad \forall i > 0.$$

$$\overset{\text{Serre}}{\underset{\text{duality}}{\Longleftrightarrow}} \quad H^i(X, \mathcal{L}^{-1}) = 0, \quad \forall i < n.$$

A) Fujita vanishing.

Thm. A ample, $\mathcal{F}$ coherent.

$\Rightarrow \exists\, m = m(A, \mathcal{F})$ s.t. $H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mA + P)) = 0$ for all $i > 0$, $m \geq m(\mathcal{F}, H)$.

Application - (Cohomology of nef line bundles)

$0 \to \mathcal{F}(mD) \to \mathcal{F}(mD + H) \to \mathcal{F}(mD + H) \otimes \mathcal{O}_H \to 0$

$h^i(X, \mathcal{F}(mD)) = O(m^{n-i})$.

$\Rightarrow$ D nef, then

$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} \cdot m^n + O(m^{n-1})$

C) Application: ~~base point freeness of~~ linear system on curves.

Prop -

Let D be a Cartier divisor on a nonsing. irreducible proj. curve C. $d = \deg D$.

1) If $d \geq 2g$, the $\mathrm{Bs}|D| = \phi$.

2) If $d \geq 2g+1$, the D is very ample

Proof -

1) Fix a point $x \in C$, consider

$$0 \to \mathcal{O}_C(-x) = \mathcal{I}_x \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C/\mathcal{I}_x = \mathbb{C}_x \to 0$$

$$\overset{\otimes \mathcal{O}_C(D)}{\leadsto} \quad 0 \to \mathcal{O}_C(D-x) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathbb{C}_x \otimes \mathcal{O}_C(D) \to 0$$

$d \geq 2g$, $\deg K_C = 2g-2$ $\Rightarrow$ $D_x := D - x$ with $\deg(D_x) \geq 2g-1$

i.e. $\deg(D_x - K_C) > 0$ and hence $D_x - K_C$ ample

$\Rightarrow h^1(C, \mathcal{O}_C(D-x)) = 0$. i.e. $H^0(C, \mathcal{O}_C(D)) \twoheadrightarrow H^0(x, \mathbb{C}_x \otimes \mathcal{O}_C(D))$

$\Rightarrow \exists\, s \in H^0(C, \mathcal{O}_C(D))$ s.t. $\underset{0}{s(x) \neq} \in H^0(x, \mathbb{C}_x \otimes \mathcal{O}_C(D))$.

2) Choose $x, x' \in C$ (maybe $x = x'$).

$$0 \to \mathcal{O}_C(-x-x') = \mathcal{I} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C/\mathcal{I} = \mathcal{Q} \to 0$$

$\otimes \mathcal{O}_C(D)$ $\rightsquigarrow$

$$0 \to \mathcal{O}_C(D-x-x') \longrightarrow \mathcal{O}_C(D) \longrightarrow (\mathcal{Q} \otimes \mathcal{O}_C(D)) \to 0.$$

$\deg(D) = d \geq 2g+1 \Rightarrow H^1(C, \mathcal{O}_C(D-x-x')) = 0$

$\Rightarrow H^0(C, \mathcal{O}_C(D)) \twoheadrightarrow H^0(\{x\}\cup\{x'\}, \mathcal{Q} \otimes \mathcal{O}_C(D))$.

$\Rightarrow \exists\, s \in H^0(C, \mathcal{O}_C(D))$ s.t. $\begin{cases} s(x) \neq 0 \text{ and } s(x') = 0 & x \neq x' \\ s(x) = v \in \mathfrak{m}_x/\mathfrak{m}_x^2, & x = x'. \text{ given } v \end{cases}$

$\mathfrak{m}_x/\mathfrak{m}_x^2 = \Omega_{C,x}$.

§2. Multiplier Ideals.

Question - Can Kodaira's vanishing thm be generalised to line bundles with weaker positivity, e.g. nef, big, pseff ?

In general, the answer is No! For example, E = an elliptic curve.

Then $H^1(E, \mathcal{O}_E) \cong H^0(E, \omega_E) = H^0(E, \mathcal{O}_E) = \mathbb{C} \neq 0$.

($\mathcal{O}_E$ nef. ; $\cong$ Serre duality.)

A) Nadel vanishing Thm

Thm - (Nadel)

Let X be a nonsingular irreducible proj. variety of dim n. L a Cartier divisor on X and $D \geq 0$ a $\mathbb{Q}$-divisor s.t. $L-D$ is nef and big. Then $\exists$ an ideal sheaf $\mathcal{J}(D)$ depending on D s.t.

$$H^i(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{J}(D)) = 0, \quad \forall i > 0$$

Remark -

1) $\mathcal{J}(D)$ is exactly the multiplier ideal sheaf associated to D

2) As big = ample + effective, the ideal sheaf can be viewed as the correction term induced by the effective ~~part~~ / negative part.

B) Multiplier ideals

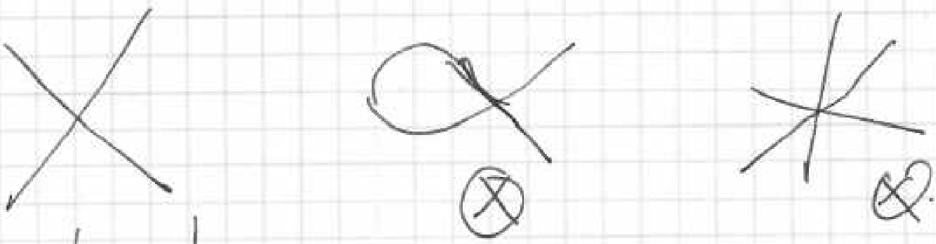
Def. - (rounds)

Let $D = \sum a_i D_i$ be a Weil $\mathbb{F}$-divisor on an irreducible variety X.

1) The round-up $\lceil D \rceil$ is $\sum \lceil a_i \rceil D_i$
2) The round-down $\lfloor D \rfloor$ is $\sum \lfloor a_i \rfloor a_i$ (or $[D]$, also called integral part)
3) The fractional part $\{D\}$ is $D - \lfloor D \rfloor$.

Def. - (Normal crossing) $a_i \neq 0$

Let X be a nonsing irreducible proj. variety. A divisor $D = \sum a_i D_i$ has simple normal crossings (or D is a SNC divisor) if each D_i is nonsingular and D is defined in an analytic nbhd of any points by local analytic coordinates of type $z_1 \cdots z_k = 0$ for some $k \le n$.



2 dim'l.

3-dim'l

Thm - (Hironaka's log resolution)

Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf on an irreducible variety X. Then $\exists$ a resolution $\mu : X' \longrightarrow X$ s.t.

1) X' nonsing
2) μ is birational, projective.

~~3) $\mu^{-1}\mathcal{I} \subseteq \mathcal{O}_{X'}$ is invertible, i.e. $\exists$~~

2) $\exists$ a Cartier divisor F s.t. $\mu^{-1}I = \mathcal{O}_{X'}(-F) \subseteq \mathcal{O}_{X'}$ and $F+E$ is SNC, where E ~~is~~ any exceptional divisor on X'. i.e. $\mu_* E = 0$.

Remark

$\mu^{-1}I$ = ideals of $\mathcal{O}_{X'}$ generated $s \circ \mu \in \mathcal{O}_{X'}$, where $s \in I \subseteq \mathcal{O}_X$.

Def.- (relative canonical divisor).

Let X be a ~~irreducible~~ nonsingular irreducible proj. variety. $\mu : X' \to X$ a birational ~~projective~~ morphism with X' nonsing. projective.

Choose canonical divisors $K_{X'}$ and K_X on X' and X respectively s.t. $\mu_* K_{X'} = K_X$ i.e. $K_{X'}\big|_{\mu^{-1}(U)} = K_X\big|_U$

where $U \subseteq X$ open subset s.t. $\operatorname{codim}(X \setminus U) \geq 2$ and $\mu^{-1}(U) \to U$ an isom.

The <u>relative canonical divisor</u> $K_{X'/X}$ is $K_{X'} - \mu^* K_X$.

Remark - (Exercise)

$\operatorname{Supp}(K_{X'/X})$ is μ-exceptional and hence $K_{X'/X}$ is independent of the choice of K_X and hence unique. Moreover, $\mu_* \mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X$.

Def.-

Let X be a nonsing. irreducible proj. variety. $D \geq 0$ $\mathbb{Q}$-divisor on X.

Fix $\mu : X' \to X$ a log-resolution of $\mathcal{O}_X(-D) = I_D \subseteq \mathcal{O}_X$. Then the <u>multiplier ideal sheaf</u> $\mathcal{J}(D) = \mathcal{J}(X, D)$ associated to D is defined as

$$\mathcal{J}(D) = \mu_* \mathcal{O}_{X'}\big(K_{X'/X} - \underbrace{\lfloor \mu^* D \rfloor}_{\geq 0}\big) \subseteq \mathcal{O}_X.$$

Remark -

$\mathcal{J}(D)$ is independent of the resolution μ.

c) Analytic meaning of $\mathcal{J}(X,D)$.

Let X be a nonsingular irreducible proj. variety of dimension n. $D = \sum a_i D_i \geq 0$ a SNC $\mathbb{Q}$-divisor with $a_i > 0$. ~~$x \in X$~~ a point.

~~1) $x \notin \mathrm{Supp}(D) = \cup D_i$~~

~~then~~

Then $\mathcal{J}(X,D) = \mathcal{O}_X(-[D]) \subseteq \mathcal{O}_X$.

1) $x \notin \mathrm{Supp}(D)$, then $\mathcal{J}(X,D)_x = \mathcal{O}_{X,x}$.

2) $x \in \mathrm{Supp}(D)$, then $\sum D_i = \{z_1 \cdots z_k = 0\}$ in an analytic nbhd of x.

$$f \in \mathcal{J}(X,D)_x \iff \deg_{z_i}(f) \geq [a_i].$$

$$\iff \int_{\underbrace{V(x)}_{\text{small nbhd of } x}} \frac{|f|^2}{|z_1|^{2a_1} \cdots |z_k|^{2a_k}} \, d\mu < +\infty.$$

$$\iff \int_{V(x)} |f|^2 e^{-2\varphi_D} \, d\mu < +\infty.$$

$\varphi_D \big|_{V(x)} = \sum a_i \log |z_i|$ Recall $\int |z|^{2(\alpha - a)} \, d\mu < +\infty$

$\iff \alpha - a > -1$.

Def: (X, D) a pair with $\begin{cases} X \text{ nonsing. proj.} \\ D \geq 0 \ \mathbb{Q}\text{-divisors.} \end{cases}$

Then (X,D) is called $\begin{cases} \text{klt} & \text{if } \mathcal{J}(X,D) = \mathcal{O}_X. \\ \text{lc} & \text{if } \mathcal{J}(X,(1-\varepsilon)D) = \mathcal{O}_X, \ \forall\, 0 < \varepsilon < 1, \ \varepsilon \in \mathbb{Q} \end{cases}$

Remark. the definition can be easily generalised ~~to $\mathbb{R}$-divisors~~ and to the case.

irreducible normal proj. variety with $K_X + D$ ~~Cartier~~ $\mathbb{R}$-divisors

(because $\mu^*(K_X + D)$ is well-defined).