Exercise sheet 6

Exercise 1 (Invariants of projective varieties)**.** Let X be a nonsingular irreducible projective variety.

(1) (Euler characteristic) Let $\mathscr{F}$ be a coherent sheaf on X. The *Euler characteristic* of $\mathscr{F}$ is defined as
$$\chi(\mathscr{F}) \coloneqq \sum(-1)^i h^i(X, \mathscr{F}).$$
Prove that if $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{Q} \to 0$ is a short exact sequence of coherent sheaves on X, then we have
$$\chi(\mathscr{E}) = \chi(\mathscr{F}) + \chi(\mathscr{Q}).$$

(2) The number $\chi(X, \mathscr{O}_X)$ is called the *Euler characteristic of* X. Determine the Euler characteristic of $\mathbb{P}^n$ and a nonsingular hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d.

(3) (Geometric genus) The *geometric genus* $p_g(X)$ *of* X is defined as $h^0(X, \omega_X)$. Determine the geometric genus of a nonsingular hypersurface in $\mathbb{P}^{n+1}$ of degree d.

(4) (Arithmetic genus) The *arithmetic geneus* $p_a(X)$ *of* X is defined as $(-1)^n(\chi(X, \mathscr{O}_X) - 1)$, where $n = \dim(X)$. Prove that if $\dim(X) = 1$, then we have
$$p_a(X) = h^1(X, \mathscr{O}_X).$$
Derive that we have $p_a(X) = p_g(X)$ if $\dim(X) = 1$.

(5) (Genus-degree formula) Let $C \subset \mathbb{P}^2$ be a nonsingular curve of degree d. Prove that we have
$$p_g(C) = p_a(C) = \frac{(d-1)(d-2)}{2}.$$
In particular, if C and C' are two nonsingular curves in $\mathbb{P}^2$ with degree $d \neq d' \geq 2$, then C is not isomorphic to C'.

(6) Show that the conic C in $\mathbb{P}^2$ defined as $X_0^2 + X_1^2 + X_3^2 = 0$ is isomorphic to $\mathbb{P}^1$.

(7) (Irregularity) The *irregularity* $q(X)$ *of* X is defined as $h^1(X, \mathscr{O}_X)$. Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of dimension $n \geq 2$. Prove that X is regular, i.e., $q(X) = 0$.

Exercise 2 (Blow-up of quadratic cone)**.** Let $X \subset \mathbb{A}_k^3$ be the quadratic cone defined by $X_1X_2 = X_3^2$. Let $\pi : V \to X$ be the blow-up of $\mathbb{A}_k^3$ with centre in the origin, and X' the closure of $\pi^{-1}(X \setminus \{0\})$; i.e. X' is the blow-up of X along 0.

(1) Prove that X' is nonsingular.

(2) Prove that the inverse image of the origin under $X' \to X$ is isomorphic to $\mathbb{P}^1$.

Exercise 3 (Du Val singularity of type D_4)**.** Let $X \subset \mathbb{A}_k^3$ be the affine subvariety defined by the equation $X_1^2 + X_2^3 + X_3^3 = 0$.

(1) Prove that the origin is the only singular point of X.

(2) Determine the blow-up $\sigma : X_1 \to X$ of X along 0.

(3) Prove that there are three singular points of X_1.

(4) Let $\pi : X_2 \to X_1$ be the blow-up of X_1 along the three singular points of X_1. Prove that X_2 is nonsingular.

(5) Draw the dual graph of the composition $f : X_2 \to X_1 \to X$; that is, a graph whose vertexes corresponds to the irreducible components of $f^{-1}(0)$ and two vertexes are joined by a line if they intersects each other.

Exercise 4 (Higher direct image)**.** In this exercise, we aim to discuss the cohomologies of coherent sheaves under base change. Let $f : X \to Y$ be a morphism between varieties. Let $\mathscr{F}$ be a quasi-coherent sheaf on X.

(1) ([Har77, III, Proposition 8.1]) For each $i \geq 0$, the i-th higher direct image $R^i f_* \mathscr{F}$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)})$$

on Y.

(2) The morphism $f : X \to Y$ is called *affine* if for every affine open subset V of Y, the inverse image $f^{-1}(V)$ is affine. Prove that if $f : X \to Y$ is affine, then $R^i f_* \mathscr{F} = 0$ for $i > 0$. (Hint: use (1) + Theorem A and B for affine varieties.)

(3) (Leray's spectral sequence, [God58, II, Théorème 4.17.1]) If $R^i f_* \mathscr{F} = 0$ for any $i > 0$, then we have a canonical isomorphism

$$H^q(Y, f_* \mathscr{F}) \cong H^q(X, \mathscr{F}).$$

Roughly speaking, the higher direct image $R^i f_* \mathscr{F}$ represents "*the cohomologies of $\mathscr{F}$ along its fibres*".

(4) Use (2) and (3) to derive that if $f : X \to Y$ is affine, then $H^q(X, \mathscr{F}) \cong H^q(Y, f_* \mathscr{F})$ for all $q \geq 0$.

(5) (Projective formula) Let $\mathscr{E}$ be a locally free sheaf over Y. Then for all $q \geq 0$ we have a canonical isomorphism

$$R^i f_*(\mathscr{F} \otimes f^* \mathscr{E}) \cong R^i f_* \mathscr{F} \otimes \mathscr{E}.$$

(6) Assume that $R^i f_* \mathscr{O}_X = 0$ for all $i > 0$. For any locally free sheaf $\mathscr{E}$ over Y and $q \geq 0$, prove that we have a canonical isomorphism

$$H^q(X, f^* \mathscr{E}) \cong H^q(Y, f_* \mathscr{O}_X \otimes \mathscr{E}).$$

Reference

[God58] Roger Godement. *Topologie algébrique et théorie des faisceaux.* Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1252. Hermann, Paris, 1958. Publ. Math. Univ. Strasbourg. No. 13.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.