Exercise sheet 5

Exercise 1 (Stalks of hom-sheaf)**.** Let $X = [0,1]$ with cofinite topology. Let $\underline{\mathbb{Z}}_X$ be the constant sheaf over X with respect to $\mathbb{Z}$.

(1) Prove that $\underline{\mathbb{Z}}_X$ is indeed a sheaf over X.

(2) Let $\mathcal{F}$ be the skycraper sheaf $\mathbb{Z}$ at $0 \in X$ (see Exercise 5 in Exercise sheet 1). Show that $\mathcal{F}$ admits a natural $\underline{\mathbb{Z}}_X$-module structure, i.e. $\mathcal{F}$ is a sheaf of $\underline{\mathbb{Z}}_X$-modules.

(3) Let $U \subset X$ be a non-empty open subset of X. Show that we have

$$\mathrm{Hom}_{\underline{\mathbb{Z}}_X}(\mathcal{F}|_U, \underline{\mathbb{Z}}_X|_U) = 0.$$

(4) Determine the stalks of $\mathcal{F}$ and $\underline{\mathbb{Z}}_X$ at 0.

(5) Determine the group $\mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{F}_0, \underline{\mathbb{Z}}_{X,0})$. In particular, prove that the natural homomorphism

$$\mathcal{H}\mathrm{om}_{\underline{\mathbb{Z}}_X}(\mathcal{F}, \underline{\mathbb{Z}}_X)_0 \to \mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{F}_0, \underline{\mathbb{Z}}_{X,0})$$

is not surjective.

(6) Let $U = X \setminus \{0\}$ and let $\underline{\mathbb{Z}}_U$ be the constant sheaf over U with respect to $\mathbb{Z}$. Denote by $\mathcal{G}$ the sheaf obtained by extending $\underline{\mathbb{Z}}_U$ by zero (see Exercise 7 in Exercise sheet 1). Prove that $\mathcal{G}$ is a sheaf of $\underline{\mathbb{Z}}_X$-modules.

(7) Determine the stalk $\mathcal{G}_0$ and derive the following

$$\mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{G}_0, \mathcal{G}_0) = 0.$$

(8) Let $V \subset X$ be a non-empty open subset of X. Show that there exists a natural inclusion

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(\mathcal{G}|_V, \mathcal{G}|_V).$$

(9) Show that there exists a natural inclusion

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \hookrightarrow \mathcal{H}\mathrm{om}_{\underline{\mathbb{Z}}_X}(\mathcal{G}, \mathcal{G})_0.$$

In particular, derive that the natural homomorphism

$$\mathcal{H}\mathrm{om}_{\underline{\mathbb{Z}}_X}(\mathcal{G}, \mathcal{G})_0 \to \mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{G}_0, \mathcal{G}_0)$$

is not injective.

Exercise 2 (Twisting sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$)**.** Let $d \in \mathbb{Z}$ be an integer. In this exercise, we aim to prove that the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ corresponds exactly the $\mathbb{O}_{\mathbb{P}^n}$-module defined by the lifted module $S(d)$, where $S = k[X_0, \ldots, X_n]$.

(1) Let $0 \neq F \in S$ be a non-zero homogeneous element. Write down the elements in the group $\Gamma(D(F), \widetilde{S(d)})$.

(2) Let $U_i \subset \mathbb{P}^n$ be the standard affine open subset $D(X_i) \subset \mathbb{P}^n$ and let $D(F) \subset U_i$ be a standard open subset contained in U_i. For any element $s \in \Gamma(D(F), \widetilde{S(d)})$, show that $s \cdot X_i^{-d}$ is a regular function on $D(F)$.

(3) For each i, we consider the following homomorphism of sheaves

$$\varphi_i : \widetilde{S(d)}|_{U_i} \to \mathscr{O}_{\mathbb{P}^n}|_{U_i}$$

which sends an element s to $X_i^{-d}s$. Show that φ_i is an isomorphism of $\mathscr{O}_{\mathbb{P}^n}$-modules.

(4) Write out the transition functions for $\mathcal{O}_{\mathbb{P}^n}(d)$.

(5) Show that the isomorphisms $\{\varphi_i\}_{0 \leq i \leq n}$ induces an isomorphism of $\widetilde{S(d)}$ and $\mathcal{O}_{\mathbb{P}^n}(d)$.

Exercise 3 (Ideal sheaves and conormal sheaves)**.** Let X be an algebraic variety and let $Y \subset X$ be a closed subvariety. Denote by $\mathcal{I}_Y$ the ideal sheaf of Y and by i the natural inclusion $Y \hookrightarrow X$.

(1) Assume that X is affine with $A = \Gamma(X, \mathcal{O}_X)$. Let I be the ideal of Y.
   (a) Show that we have a natural isomorphism $\mathcal{I}_Y \cong \widetilde{I}$ of $\mathcal{O}_X$-modules.
   (b) Show that we have a natural isomorphism $\mathcal{I}_Y/\mathcal{I}_Y^2 \cong \widetilde{I/I^2}$ as $\mathcal{O}_X$-modules.
   (c) Show that we have a natural isomorphism $i^*\mathcal{I}_Y \cong \mathcal{N}^*_{Y/X}$ of $\mathcal{O}_Y$-modules.

(2) Show that we have a natural isomorphism $i^*\mathcal{I}_Y = \mathcal{N}^*_{Y/X}$.

(3) Let $Y \subset X = \mathbb{P}^n$ be a hypersurface of degree d. Using the statement (2) above to show that $\mathcal{N}^*_{Y/X} \cong i^*\mathcal{O}_{\mathbb{P}^n}(-d)$.

(4) Use the following cotangent sequence to determine the canonical sheaf Ω_Y for Y being a nonsingular hypersurface of degree d in $\mathbb{P}^n$

$$0 \to \mathcal{N}^*_{Y/X} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_Y \to 0.$$

Exercise 4 (Cohomologies of cotangent sheaves)**.** In this exercise, we want to compute the cohomologies of the cotangent sheaves of certain nonsingular varieties.

(1) Use the Čech cohomology to show that for a family $\{\mathcal{F}_i\}_{i\in I}$ of quasi-coherent sheaves on an algebraic variety X, we have a natural isomorphism

$$H^q(X, \bigoplus_{i\in I} \mathcal{F}_i) \cong \bigoplus_{i\in I} H^q(X, \mathcal{F}_i).$$

(Hint: use the fact that X is quasi-compact and find a suitable finite affine open covering of X.)

(2) Now we want to compute the cohomologies of $\Omega_{\mathbb{P}^n}$ using the following Euler sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

   (a) Determine the cohomologies of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$ using (1).
   (b) Determine the cohomologies of $\mathcal{O}_{\mathbb{P}^n}$.
   (c) Use the Euler sequence to derive a long exact sequence to determine the cohomologies of $\Omega_{\mathbb{P}^n}$.

(3) Let $i : X \hookrightarrow \mathbb{P}^{n+1}$ be a nonsingular quadric hypersurface. In the following we want to compute the cohomologies of Ω_X. In the proof, you can use the following fact: let $j : Z \hookrightarrow Y$ be a closed variety in an algebraic variety Y and let $\mathcal{F}$ be a coherent sheaf, then we have a natural isomorphism, for each $q \geq 0$,

$$H^q(Z, j^*\mathcal{F}) \cong H^q(Y, \mathcal{F} \otimes j_*\mathcal{O}_Z).$$

   (a) Show that the following sequence of coherent sheaves is exact

$$0 \to \Omega_{\mathbb{P}^n}(-2) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(-2) \to 0.$$

   Then use it to determine the cohomologies of $\Omega_{\mathbb{P}^n}(-2)$.
   (b) Show that the following sequence of coherent sheaves is exact

$$0 \to \Omega_{\mathbb{P}^n} \otimes \mathcal{I}_X \to \Omega_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n} \otimes i_*\mathcal{O}_X \to 0.$$

   Then use it to determine the cohomologies of $i^*\Omega_{\mathbb{P}^n}$.
   (c) Use the following conormal sequence to determine the cohomologies $H^q(X, \Omega_X)$ for $q \leq n-2$

$$0 \to \mathcal{N}^*_{X/\mathbb{P}^n} \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0.$$