University of Chinese Academy of Sciences
Pure Mathematics

Fall 2022
Algebraic Geometry I & II

Exercise sheet 4

Exercise 1 ($\mathbb{Q}$-Cartier divisor which is not Cartier)**.** Let $X = \{X_1X_2 = X_3^2\} \subset \mathbb{A}_k^3$ be the variety given in the course. Let D be the prime divisor in X defined by the following equation $X_1 = X_3 = 0$.

(1) Find the singular locus of X.
(2) Let $U = D(X_2) \subset X$ be the standard open subset of X defined as $X_2 \neq 0$. Find the generators of the ideal of $D \cap D(X_2)$ in $\Gamma(D(X_2), \mathcal{O}_X)$.
(3) Show that the Weil divisor $2D$ is Cartier.
(4) Let $Y \subset \mathbb{A}_k^3$ be the variety defined by the following equation $Y_2 = Y_3^2$. Show that Y is nonsingular.
(5) Consider the morphism $\Phi : \mathbb{A}_k^3 \to \mathbb{A}_k^3$ which sends the point (x_1, x_2, x_3) to the point (x_1, x_1x_2, x_1x_3). Prove that the restriction $f := \Phi|_Y$ of Φ to Y is a birational morphism from Y to X.
(6) Let $E = f^{-1}(0)$ be the preimage of the original point in Y. Show that E is a prime divisor in Y and find the generators of the ideal of E in $\Gamma(Y, \mathcal{O}_Y)$.
(7) Show that $f|_{Y\setminus E} : Y \setminus E \to X \setminus D$ is an isomorphism.
(8) Calculate the pull-back $f^*(2D)$ and prove that D is not Cartier.
(9) Let $D' \subset X$ be the prime divisor in X defined by the equation $X_2 = X_3 = 0$. Show that $2D'$ is Cartier.
(10) Calculate the pull-back $f^*(2D')$ and then prove that D' is not Cartier.
(11) Prove that $D + D'$ is Cartier.

Exercise 2 (Restriction to open subsets)**.** Let X be an irreducible normal variety. Let $Z \subset X$ be a proper closed subset of X and let $U = X \setminus Z$.

(1) Show that there is a surjective homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined as following:
$$D = \sum n_i D_i \mapsto \sum n_i (D_i \cap U),$$
where D_i is a prime divisor and we igonore those $D_i \cap U$ which are empty.
(2) If Z has codimenmsion at least two in X, show that $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism.
(3) If Z is a prime divisor, prove that the following sequence is exact:
$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$
where the first map is defined by $1 \mapsto 1 \cdot Z$.
(4) We remark that if X is an irreducible normal affine variety, then $\mathrm{Cl}(X) = 0$ if and only if $\Gamma(X, \mathcal{O}_X)$ is a UFD, see for instance [Har77, II, Proposition 6.2].
  (a) Use the remark above to show that $\mathrm{Cl}(\mathbb{A}_k^n) = 0$.
  (b) Let D be the prime divisor in $\mathbb{P}^n(k)$ defined by $X_0 = 0$. Prove that D is not a principal divisor. (Hint: the regular functions on $\mathbb{P}^n(k)$ are constants.)
  (c) Show that $\mathrm{Cl}(\mathbb{P}^(k)) = \mathrm{CaCl}(\mathbb{P}^n(k)) \cong \mathbb{Z}$ and D is a generator for it.
  (d) Prove that $\mathrm{Pic}(\mathbb{P}^n(k)) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}^n(k)}(1)$.

Exercise 3 (Class group of affine varieties)**.** In this exercise we aim to show that the variety X defined in Exercise 1 is $\mathbb{Q}$-factorial and determine its class group.

(1) Let $X = \{X_1X_2 = X_3^2\} \subset \mathbb{A}_k^3$ be the variety as in Exercise 1 and we follow the same notations as in Exercise 1.

(2) Show that we have $\Gamma(U, \mathcal{O}_X) = k[X_1, X_1^{-1}, X_3]$, where $U = X \setminus D$.
(3) Show that U is a nonsingular irreducible affine variety and $\Gamma(U, \mathcal{O}_U)$ is a UFD.
(4) Use Exercise 2 to show that $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and D is a generator for it.
(5) Prove that X is $\mathbb{Q}$-factorial.

Exercise 4 (Blowing-up points of affine spaces)**.** For $n \geq 1$, let $X \subset \mathbb{A}_k^{n+1} \times \mathbb{P}^n(k)$ be the closed subvariety defined by the following equation

$$X_i Y_j = X_j Y_i, \quad 0 \leq i, j \leq n.$$

where $(X_0, \ldots, X_n)$ is the coordinate of $\mathbb{A}_k^{n+1}$ and $[Y_0 : \cdots : Y_n]$ is the homogeneous coordinate of $\mathbb{P}^n(k)$. Denote by $\pi : X \to \mathbb{A}_k^{n+1}$ the first projection. Let $0 \in \mathbb{A}_k^{n+1}$ be the original point and denote by E the preimage $\pi^{-1}(0)$.

(1) Prove that X is irreducible and nonsingular.
(2) Prove that $E \cong \mathbb{P}^n(k)$.
(3) Prove that $\pi|_{X \setminus E} : X \setminus E \to \mathbb{A}_k^{n+1} \setminus \{0\}$ is an isomorphism.
(4) For any integer $a \in \mathbb{Z}$, prove that the divisor aE is principal if and only if $a = 0$. (Hint: otherwise, show that aE is given by $\mathrm{div}(f \circ \pi)$, where f is a regular function on $\mathbb{A}_k^{n+1}$ nonwhere vanishing).
(5) Use Exercise 2 to determine the class group $\mathrm{Cl}(X)$.

The morphism $\pi : X \to \mathbb{A}_k^n$ is called the *blowing-up of* $\mathbb{A}_k^n$ *along* 0.

Exercise 5 (Picard group of affine varieties)**.** Let $Y \subset \mathbb{P}^n(k)$ be an irreducible hypersurface and let F be the generator of the homogeneous ideal of Y. Denote by d the degree of F. Let $X = \mathbb{P}^n(k) \setminus Y$.

(1) Show that X is an irreducible nonsingular affine variety.
(2) Use Exercise 2 to show that $\mathrm{Cl}(X) = \mathrm{CaCl}(X) \cong \mathbb{Z}/d\mathbb{Z}$ and H is a generator for it, where H is a hyperplane section; that is, H is defined by a linear form.
(3) Show that X is not isomorphic to an affine space.

Exercise 6 (Automorphism group of projective spaces)**.** Recall that $\mathrm{GL}(n+1, k)$ is the group of all invertible $(n+1) \times (n+1)$-matrices over k with the operation of matrices multiplication.

(1) Show that for every element $A \in \mathrm{GL}(n+1, k)$ induces a natural isomorphism of $\mathbb{P}^n(k)$.
(2) Show that the kernel of the action of $\mathrm{GL}(n+1, k)$ on $\mathbb{P}^n(k)$ is the subgroup $\{cI \mid c \in k^*\}$ of central scalar matrices. Denote by $\mathrm{PGL}(n+1, k)$ the quotient group. We call $\mathrm{PGL}(n+1, k)$ the *projective general linear group*.
(3) Let $g : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ be an automorphism of $\mathbb{P}^n(k)$. Show that g induces a linear isomorphism of the k-vector space $H^0(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)}(1))$.
(4) An automorphism of $\mathbb{P}^n(k)$ is called a *projective transformation* if it is given by an element in $\mathrm{PGL}(n+1, k)$. Show that every automorphism of $\mathbb{P}^n(k)$ is a projective transformation.

Reference

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.