Exercise sheet 3

Exercise 1 (Singular points of projective hypersurfaces)**.** Let k be an algebraically closed field of characteristic zero. Let $\mathbb{P}^n(k)$ be the n-dimensional projective space over k. Recall that a *hypersurface* of $\mathbb{P}^n(k)$ is a projective subvariety of $\mathbb{P}^n(k)$ defined by a non-zero homogeneous polynomial. Moreover, given a hypersurface X in $\mathbb{P}^n(k)$, then there exists a unique reduced polynomial F such that the homogenous ideal of X is generated by F. Then X is said to be defined by the polynomial F.

(1) Prove that the singular points of a hypersurface $X \subset \mathbb{P}^n(k)$, which is defined by a homogeneous polynomial $F(x_0, \cdots, x_n) = 0$, are determined by the system of equations
$$F(x_0, \cdots, x_n) = 0 \quad \text{and} \quad \tfrac{\partial F}{\partial X_i}(x_0, \cdots, x_n) = 0 \text{ for } i = 0, \cdots, n.$$
(2) Prove that we have the following equality, which is known as *Euler's Theorem.*
$$\deg(F) \cdot F = \sum_{i=0}^{n} X_i \frac{\partial F}{\partial X_i}.$$
(3) Determine the singular points of the Steiner surface in $\mathbb{P}^3(k)$:
$$x_1^2 x_2^2 + x_2^2 x_0^2 + x_0^2 x_1^2 - x_0 x_1 x_2 x_3 = 0.$$
(4) Prove that if a hypersurface $X \subset \mathbb{P}^n(k)$ contains a linear subspace L of dimension $r \geq n/2$, then X is singular. (Hint: choose the coordinate system so that L is given by $x_{r+1} = \cdots = x_n = 0$, write out the equation of X and look for singular points contained in L.)

(5) Let $p \in \mathbb{P}^n(k)$ be a point and let $L_1, \ldots, L_n$ be n linear forms in $k[x_0, \ldots, x_n]$ such that $V(L_1, \ldots, L_n) = \{p\}$. Let $\pi_p : \mathbb{P}^n(k) \dashrightarrow \mathbb{P}^{n-1}(k)$ be the rational map defined as following:
$$[x_0 : \cdots : x_n] \mapsto [L_1(x_0, \ldots, x_n) : \cdots : L_n(x_0, \ldots, x_n)].$$
Show that π_p is a well-defined morphism over $\mathbb{P}^n(k) \setminus \{p\}$.

(6) Let $p \in \mathbb{P}^n(k)$ be a point. A *cone over* p is the closure of the preimage $\pi_p^{-1}(Y)$ for a projective subvariety $Y \subset \mathbb{P}^{n-1}(k)$. Prove that a hypersurface of degree two with a singular point is a cone. Here the degree of hypersurface is defined as the degree of the defining polynomial. (Hint: consider the projection from a singular point).

(7) Let X be an irreducible hypersurface of degree 3. Assume that the singular locus of X contains two distinct points p and q. Prove that the line joining p and q is contained in X. Here a line means a projective subspace of dimension one in $\mathbb{P}^n(k)$.

Exercise 2 (Projective tangent spaces)**.** Let $X \subset \mathbb{P}^n(k)$ be an irreducible projective variety and let $p \in X$ be a point. Show that the following definitions of the "**projective tangent space**" of X at p are equivalent:

(1) The closure in $\mathbb{P}^n(k)$ of the tangent space to the affine variety $X \cap U_i$ at p, where U_i is any standard affine chart containing p.

(2) The projective linear subspace corresponding to the subspace of k^{n+1}, which is the kernel of the $r \times (n+1)$ scalar matrix
$$J = \left(\frac{\partial F_i}{\partial X_j}(x_0, \ldots, x_n) \right),$$

where $\{F_1, \ldots, F_r\}$ is a family of homogeneous generators of the homogeneous ideal $V(X)$ and $(x_0, \cdots, x_n) \in k^{n+1}$ is an arbitrary point representing p.

(3) The projective linear subspace corresponding to the linear subspace $T_{\widetilde{p}}\widetilde{X}$ of k^{n+1}, where $\widetilde{X} \subset k^{n+1}$ is the affine cone of X and $\widetilde{p} \in \widetilde{X}$ is any point representing p.

Exercise 3 (Dual varieties)**.** Let $X \subset \mathbb{P}^n(k)$ be an irreducible non-singular projective variety. Let $\check{\mathbb{P}}^n(k)$ be the dual space of $\mathbb{P}^n(k)$; that is, the projection space associated to the dual vector space $(k^{n+1})^*$. Given a point $\xi \in \check{\mathbb{P}}^n(k)$, we denote by $H_\xi \subset \mathbb{P}^n(k)$ the hyperplane defined by ξ; that is, the $(n-1)$-dimensional projective space associated to the kernel of ξ in k^{n+1}.

(1) Consider the following set $I_X \subset \mathbb{P}^n(k) \times \check{\mathbb{P}}^n(k)$, which is called the *conormal variety* of X:

$$I_X :\, = \{(x, \xi) \in \mathbb{P}^n(k) \times \check{\mathbb{P}}^n(k) \,|\, x \in X, \mathbb{T}_x X \subset H_\xi\},$$

where $\mathbb{T}_x X$ is the projective tangent space of X at x. Show that I_X is an irreducible closed subset. The *dual variety* $\check{X} \subset \check{\mathbb{P}}^n(k)$ is defined as the image of I_X under the second projection. Show that $\check{X}$ is closed.

(2) Let $F(X_0, X_1, X_2) = 0$ be the equation of an irreducible non-singular curve $X \subset \mathbb{P}^2$. Consider the rational map $\varphi : X \dashrightarrow \mathbb{P}^2$ given by the formulas

$$[x_0, x_1, x_2] \mapsto [\frac{\partial F}{\partial X_0}(x_0, x_1, x_2) : \frac{\partial F}{\partial X_1}(x_0, x_1, x_2) : \frac{\partial F}{\partial X_2}(x_0, x_1, x_2)].$$

Show that the image of φ is isomorphic to the dual variety of X.

(3) In (2), prove that $\varphi(X)$ is a point if and only if X is a line.

(4) In (2), prove that if X is a conic, then so is $\varphi(X)$.

(5) Find the dual curve of $X_0^3 + X_1^3 + X_2^3 = 0$ and determine its singular points.