Exercise sheet 2

Exercise 1 (Maximal spectrum and Zariski topology)**.** Let A be a commutative ring with identity element 1. Let $X = \mathrm{MaxSpec}(A)$ be the set of all maximal ideals in A. For an ideal $\mathfrak{a}$ of A, we define $V(\mathfrak{a})$ to be the subset of X consisting of all maximal ideals of A containing $\mathfrak{a}$.

(1) Show that $V(0) = X$ and $V(1) = \emptyset$.

(2) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, show that

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b}).$$

(3) If $\{\mathfrak{a_i}_{i\in I}\}$ is a family of ideals in A, show that

$$V\left(\sum_{i\in I} \mathfrak{a}_i\right) = \bigcap_{i\in I} V(\mathfrak{a}_i).$$

These properties show that all $V(\mathfrak{a})$ satisfy the axioms for closed subsets in a topological space. We call it the *Zariski topology* on the *maximal spectrum* $X = \mathrm{MaxSpec}(A)$ of A.

Let k be an algebraically closed field and let $S \subset \mathbb{A}^n_k$ be an affine algebraic set. Let $I(S)$ be the ideal of S and denote by A the coordinate ring $k[x_1, \ldots, x_n]/I(S)$.

(1) (Weak Nullstellensatz) Let $x \in S$ be a point and let $\mathfrak{m}_x \subset A$ be the ideal consisting of elements which vanish on x. Show that $\mathfrak{m}_x$ is a maximal ideal of A and the map $\Phi : S \to X = \mathrm{MaxSpec}(A)$ by sending x to $\mathfrak{m}_x$ is a bijection.

(2) Show that Φ is a homeomorphism with respect to the Zariski topologies on S and X.

(3) Give an example to show that Φ is not surjective if k is not algebraically closed, e.g. $k = \mathbb{R}$.

Exercise 2 (Twisted cubic)**.** Let k be an algebraically closed field. We define a morphism $\varphi : \mathbb{A}^1_k \to \mathbb{A}^3_k$ by sending t to (t, t^2, t^3), where t is the coordinate of $\mathbb{A}^k_1$. let C be the image of φ. Let (x, y, z) be the coordinates of $\mathbb{A}^3_k$.

(1) Show that $C = V(I)$, where I is the ideal

$$\left(z - xy, y^2 - xz, x^2 - y\right)$$

(2) Show that C is irreducible.

(3) Show that C is non-singular.

(4) Show that C is one-dimensional.

Let $\psi : \mathbb{A}^3_k \to \mathbb{A}^2_k$ be the morphism by sending (x, y, z) to (y, z) and denote by B the image of C under ψ. Let (u, v) be the coordinates of $\mathbb{A}^2_k$.

(1) Show that $B = V(u^3 - v^2)$.

(2) Find the Zariski tangent space of B at $(0, 0)$.

(3) Show that $(0, 0)$ is a singular point of B.

(4) Show that B is not normal.

Exercise 3 (Basic properties of separatedness)**.** Let Y be a separated prevariety.

(1) Any subvarieties of Y are separated.

(2) The intersection of two open affine subvarieties of Y is again an open affine subvariety.

(3) Let $f : X \to Y$ be a morphism from a prevariety X. Prove that the graph of f, $G(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$.

Exercise 4 (Basic properties of completeness)**.** Let X be a complete algebraic variety.

(1) Let $f : X \to Y$ be a morphism to an algebraic variety. Then $f(X)$ is complete.
(2) If Y is a complete algebraic variety. Then $X \times Y$ is again complete.
(3) If $Y \subsetneq X$ is a closed subvariety, then Y is complete.
(4) If X is affine, then $\dim(X) = 0$. In particular, the affine space $\mathbb{A}^n_k$ is not complete if $n \geq 1$.

Exercise 5 (Example of varieties which are not affine)**.** Let k be a field and let $Y = \mathbb{A}^2_k$ be the affine plane over k with coordinates x and y. Denote by X the subvariety $\mathbb{A}^2_k \setminus \{(0,0)\}$.

(1) If $k = \mathbb{R}$, show that X is an affine variety.
(2) Show that we have a natural inclusion $\Gamma(Y, \mathscr{O}_Y) \subseteq \Gamma(X, \mathscr{O}_X)$ which induces an isomorphism of field $k(X) \cong k(Y)$.
(3) For any $G \in \Gamma(X, \mathscr{O}_X) \subset k(X) = k(Y)$ viewed as an element in $k(Y)$, we define
$$I_G \coloneqq \{P \in \Gamma(Y, \mathscr{O}_Y) \mid PG \in \Gamma(Y, \mathscr{O}_Y)\}.$$
Show that I_G is an ideal of $\Gamma(Y, \mathscr{O}_Y)$ and $V(I_G) \subseteq \{(0,0)\}$.
(4) Assume that k is algebraically closed. Show that there exist two non-negative integers m and n such that $x^m G \in \Gamma(Y, \mathscr{O}_Y)$ and $y^n G \in \Gamma(Y, \mathscr{O}_Y)$.
(5) Assume that k is algebraically closed. Deduce that the inclusion $\Gamma(X, \mathscr{O}_X) \subset \Gamma(Y, \mathscr{O}_Y)$ is an equality. Show that X is not an affine variety.

Exercise 6 (Quadric surfaces)**.** Let k be a field. Consider the surface $X := \{xy = st\} \subset \mathbb{P}^3(k)$ and let $Y = \mathbb{P}^2(k)$. We will show that X and Y are birational, but they are not isomorphic.

(1) Find two disjoint lines on X.
(2) Show that S is isomorphic to $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$.
(3) Show that for any curve $C \subset Y$, the open subvariety $Y \setminus C$ is affine.
(4) Deduce that any two curves in Y intersect, and that X is not isomorphic to Y.
(5) Show that $p : X \dashrightarrow Y$ given by $[x : y : s : t] \mapsto [x : y : s]$ is birational.

Exercise 7 (Fermat cubic hypersurfaces)**.** Let k be an algebraically closed field. Let $X_n \subset \mathbb{P}^{n+1}(k)$ be the hypersurface defined by the Fermat equation
$$x_0^3 + x_1^3 + \cdots + x_n^3 = 0.$$

(1) Show that X_n is an irreducible non-singular projective variety of dimension n.
(2) Show that X_1 is not birational to $\mathbb{P}^1(k)$.
(3) Show that X_2 contains exactly 27 lines.
(4) Show that X_2 contains two disjoint lines, and deduce that X_2 is birational to $\mathbb{P}^2(k)$.