Exercise sheet 1

Exercise 1 (stalks of presheaves)**.** Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X and let $x \in X$ be a point. Show that there exists a natural abelian group struction on $\mathscr{F}_x$ such that for any open neighbourhood U of x, the restriction map $\varphi_{U,x} : \Gamma(U, \mathscr{F}) \to \mathcal{F}_x$ is a homomorphism of abelian groups

Exercise 2 (morphisms of sheaves)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of abelian groups on a topological space X and let $\varphi : \mathscr{G} \to \mathscr{F}$ be a morphism between presheaves.

(1) Show that the germ $\varphi_x : \mathscr{G}_x \to \mathscr{F}_x$ is a well-defined homomorphism of abelian groups.
(2) Assume furthermore that both $\mathscr{F}$ and $\mathscr{G}$ are sheaves. Prove that the kernel $\ker(\varphi)$ is again a sheaf.
(3) Assume furthermore that both $\mathscr{F}$ and $\mathscr{G}$ are sheaves. Prove that φ is an isomorphism if and only if $\varphi_x : \mathscr{G}_x \to \mathscr{F}_x$ is an isomorphism for any point $x \in X$.

Exercise 3 (direct limit)**.** Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. Fix a point $x \in X$.

(1) Recall that a partially ordered set I is said to be a *directed set* if for each i, j in I, there exists k such that $i \leq k$ and $j \leq k$. Let $\mathfrak{U}(x)$ be the set of open neighbourhoods of x. We say that two elements U, $V \in \mathfrak{U}(x)$ satisfy $U \leq V$ if $V \subseteq U$. Prove that $\mathfrak{U}(x)$ is a directed set.
(2) Let I be a directed set. Recall that a *direct system* over the directed set I consists of a family of abelian groups $\{G_i\}_{i\in I}$ indexed by I such that for each pair $i \leq j$, there exists a homomorphism $\mu_{ij} : G_i \to G_j$ of abelian groups such that
   (a) $\mu_{ii} = Id_{G_i}$ for each $i \in I$, and
   (b) $\mu_{jk} \circ \mu_{ij} = \mu_{ik}$ for each triple (i, j, k) such that $i \leq j \leq k$.

   Show that the set $\{\Gamma(U, \mathscr{F})\}_{U\in\mathfrak{U}(x)}$ with natural restriction maps forms a direct system.
(3) Let $\{G_i\}_{i\in I}$ be a direct system over a directed set I. Let C be the direct sum of G_i and let D be the subgroup of C generated by the elements of the the form $x_i - \mu_{ij}(x_i)$, where $x_i \in G_i$ and $i \leq j$. Then the *direct limit* $\varinjlim G_i$ of the direct system $\{G_i\}_{i\in I}$ is defined as the quotient C/D. Check that the stalk $\mathscr{F}_x$ defined in the course is exactly the direct limit
$$\varinjlim_{U\in\mathfrak{U}(x)} \Gamma(U, \mathscr{F}).$$

Exercise 4 (image of sheaves)**.** In this exercise, we construct an example $\varphi : \mathscr{G} \to \mathscr{F}$ of morphisms of sheaves such that $\text{preim}(\varphi)$ is not sheaf.

(1) Show that $\text{preim}(\varphi)$ is a presheaf.
(2) Let $X = \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$. For any open subset $U \subset X$, we define
$$\Gamma(U, \mathscr{G}) = \{s : U \to \mathbb{C} \mid s \text{ is continuous}\},$$
$$\Gamma(U, \mathscr{F}) = \{s : U \to \mathbb{C}^* \mid s \text{ is continuous}\}.$$
   Prove the $\mathscr{G}$ and $\mathscr{F}$ are sheaves on X. (Hint: the group structions on $\Gamma(U, \mathscr{F})$ and $\Gamma(U, \mathscr{G})$ are different!)
(3) For any open subset $U \subset X$, we define a map $\varphi_U : \Gamma(U, \mathscr{G}) \to \Gamma(U, \mathscr{F})$ as following:
$$\varphi_U(s) := \exp(s), \forall s \in \Gamma(U, \mathscr{G}).$$
   Show that $\varphi : \mathscr{G} \to \mathscr{F}$ is a morphism of sheaves.

(4) Let $x \in X$ be a point and let $t_x \in \mathscr{F}_x$ be a stalk. Show that there exists an element $s_x \in \mathscr{G}_x$ such that $\varphi_x(s_x) = t_x$. In particular, the morphism φ is surjective. (Hint: one may choose a small open neighbourhood U of x such that $t(U)$ is contained in a single-valued domain of $\mathrm{In}(z)$.)

(5) Show that the map $\Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{F})$ is not surjective.

This example shows why $\mathrm{preim}(\varphi) \neq \mathrm{im}(\varphi)$ in general: given a section $t \in \Gamma(U, \mathscr{F})$, it may happen that locally at the neighbourhood of every point $x \in U$, we can find a preimage of t in $\mathscr{G}$. However, these local sections of $\mathscr{G}$ can NOT be glued to be a section of $\Gamma(U, \mathscr{G})$.

e.skyscraper

Exercise 5 (Skyscraper sheaf)**.** Let X be a topological space and let A be an abelian group. Fix a point $x \in X$. For an open subset $U \subseteq X$, define

$$\Gamma(U, \mathscr{F}) = \begin{cases} A, & \text{if } x \in U \\ 0, & \text{if } x \notin U. \end{cases}$$

(1) Prove that $\mathscr{F}$ is a sheaf.

(2) Find the stalks of $\mathscr{F}$.

supportsection

Exercise 6 (Supports of sections)**.** Let $\mathscr{F}$ be a sheaf on a topological space X. Let U be an open subset of X and let $s \in \Gamma(U, \mathscr{F})$ be a section of $\mathscr{F}$ over U. The *support* of s is defined as

$$\mathrm{Supp}(s) \coloneqq \{x \in U \mid s_x \neq 0\}.$$

(1) Prove that $\mathrm{Supp}(s)$ is a closed subset of U.

(2) For s, $t \in \Gamma(U, \mathscr{F})$, we have $\mathrm{Supp}(s+t) \subset \mathrm{Supp}(s) \cup \mathrm{Supp}(t)$;

(3) If $V \subseteq U$ is an open subset, then $\mathrm{Supp}(s|_V) = \mathrm{Supp}(s) \cap V$.

(4) For a morphism $\varphi : \mathscr{F} \to \mathscr{G}$, we have $\mathrm{Supp}(\varphi_U(s)) \subset \mathrm{Supp}(s)$.

Exercise 7 (extending a sheaf by zero)**.** Let U be an open subset of a topological space X, and let $\mathscr{F}$ be a sheaf on U. Denote by $j : U \to X$ the inclusion. We define $j_!\mathscr{F}$ to be the sheaf on X associated to the presheaf $V \mapsto \Gamma(V, \mathscr{F})$ if $V \subset U$, $V \mapsto 0$ otherwise. We call $j_!\mathscr{F}$ the sheaf obtained by *extending* $\mathscr{F}$ *by zero* outside U.

(1) Find the stalks of $j_!\mathscr{F}$.

(2) Compare $j_!\mathscr{F}$ with the direct image $j_*\mathscr{F}$.

(3) Similar to the support of a section defined in Exercise 6, we can define the support of a sheaf $\mathscr{G}$ on a topological space X as

$$\mathrm{Supp}(\mathscr{G}) \coloneqq= \{x \in X \mid \mathscr{G}_x \neq 0\}.$$

Give an example to show that the support of a sheaf may be NOT a closed subset of X.