(1) 本考试为闭卷考试，不得使用任何书籍、笔记、资料、电子设备。
(2) 考试时间为 2023 年 01 月 04 号 13:30–16:00，150 分钟，总分为 100 分。
(3) 在所有的题目中，域 k 总是假设为特征为 0 的代数闭域。

Problem 1 (30 points)**.** For the following prevarieties, determine whether it is a variety or not. If your answer is no, explain your reasons. If your answer is yes, then determine whether it is an affine variety or a projective variety and explain your reasons.

(1) (5 points) $\mathbb{A}^1_k \setminus S$, where S is a finite set.

Solution. The prevariety $V := \mathbb{A}^1_k \setminus S$ is an affine variety, but not projective.

- *V is a variety.* Since $\mathbb{A}^1_k$ is separated and $V \subset \mathbb{A}^1_k$ is an open subset, thus V is also separated.
- *V is an affine variety.* Since $S \subset \mathbb{A}^1_k$ is finite, we can find a polynomial $F \in k[X] = \Gamma(\mathbb{A}^1_k, \mathscr{O}_{\mathbb{A}^1_k})$ such that $V(F) = S$, e.g. $F = 1$ if $S = \emptyset$ and $F = \prod(X - a_i)$ if $S = \{a_i\}$. Then V is the standard open subset $D(F)$ of $\mathbb{A}^1_k$, which is affine.
- *V is not projective.* Note that $\mathbb{A}^1_k$ is irreducible and so is V. In particular, if V is projective, then $\Gamma(V, \mathscr{O}_V) = k$, which is absurd as $X|_V$ is not constant.

□

(2) (5 points) $\mathbb{P}^1(k) \setminus S$, where S is a non-empty finite set.

Solution. The prevariety $V = \mathbb{P}^1(k) \setminus S$ is an affine variety, but not projective.

- Since S is not empty, one can find a point $s \in S$. Set $S' = S \setminus \{s\}$. Note that the open subvariety $\mathbb{P}^1(k) \setminus \{s\}$ of $\mathbb{P}^1(k)$ is isomorphic to $\mathbb{A}^1_k$. Hence, we have $V = \mathbb{A}^1_k \setminus S'$ and we are done by (1).

□

(3) (5 points) $V(X_0Y_0 + X_1Y_1 + X_2Y_2) \subset \mathbb{A}^3_k \times \mathbb{P}^2(k)$, where (X_0, X_1, X_2) are the coordinates of $\mathbb{A}^3_k$ and $[Y_0 : Y_1 : Y_2]$ are the homogeneous coordinates of $\mathbb{P}^2(k)$.

Solution. The prevariety V is a variety, but it is neither affine nor projective.

- *V is separated.* Note that $\mathbb{A}^3_k \times \mathbb{P}^2(k)$ is separated. Set $U_i = D(Y_i) \subset \mathbb{P}^2(k)$. Then $\{\mathbb{A}^3_k \times U_i\}_{0 \leq i \leq 2}$ form an affine open covering of $\mathbb{A}^3_k \times \mathbb{P}^2(k)$. Moreover, note that $V \cap (\mathbb{A}^3_k \times U_i)$ is closed for any $0 \leq i \leq 2$, e.g.
$$V \cap (\mathbb{A}^3_k \times U_0) = V(X_0 + X_1Y_1 + X_2Y_2),$$
where (X_0, X_1, X_2) are the coordinates of $\mathbb{A}^3_k$ and $[1 : Y_1 : Y_2]$ are the coordinates of $U_0 \cong \mathbb{U}_0$. In particular, V is a closed subvariety of $\mathbb{A}^3_k \times \mathbb{P}^2(k)$ and hence is separated.
- *V is not affine.* Consider the first projection $p : V \to \mathbb{A}^3_k$, which is clearly surjective. Then the fibre $F_0 = p^{-1}(0)$ of p over the origin is exactly $\mathbb{P}^2(k)$. In particular, F_0 is a closed projective variety of V with dimension 2. Hence, V is not affine. Otherwise, we assume to the contrary that V is affine. Then there exists a positive integer N such that $V \subset \mathbb{A}^N_k$. As F_0 has dimension 2, it follows that there exists a projection to coordinates $\pi : \mathbb{A}^N_k \to \mathbb{A}^1_k$ such that $\pi(F_0)$ is dense in $\mathbb{A}^1_k$. However, since F_0 is projective, the image $\pi(F_0)$ is also complete. Since F_0 is irreducible, the image $\pi(F_0)$ is a point, which is a contradiction as $\pi(F_0)$ is dense in $\mathbb{A}^1_k$.
- *V is not projective.* Assume to the contrary that V is projective. Then $p(V) = \mathbb{A}^3_k$ is complete, which is absurd.

□

Remark. The same argument as in the proof can be used to show that a quasi-affine variety is complete if and only if it is 0-dimensional. □

(4) (5 points) $V(X_0Y_0 + X_1Y_1 + X_2Y_2) \subset \mathbb{P}^2(k) \times \mathbb{P}^2(k)$, where $[X_0 : X_1 : X_2]$ (resp. $[Y_0 : Y_1 : Y_2]$) are the homogeneous coordinates of the first (resp. second) factor.

Solution. The prevariety V is a projective variety, but not affine

- *V is projective.* Firstly we note that $\mathbb{P}^2(k)\times\mathbb{P}^2(k)$ is projective by the Segre embedding $\mathbb{P}^2(k)\times\mathbb{P}^2(k)\hookrightarrow\mathbb{P}^8(k)$

  $$[X_0 : X_1 : X_2][Y_0 : Y_1 : Y_2] \mapsto [X_0Y_0 : X_0Y_1 : \cdots : X_2Y_1 : X_2Y_2].$$

  On the other hand, as in (3), one can easily prove that V is closed subvariety of $\mathbb{P}^2(k)\times\mathbb{P}^2(k)$ and hence also projective.
- *V is not affine.* Note that V is projective and has dimension 3 by Krull's Hauptidealsatz. Thus, as in (3), V can not be affine.

□

(5) (5 points) $\mathbb{A}^3_k\setminus V(X_1,X_2)$, where (X_1,X_2,X_3) are the coordinates of $\mathbb{A}^3_k$. (Hint: use algebraic Hartogs theorem)

Solution. The prevariety $V:=\mathbb{A}^3_k\setminus V(X_1,X_2)$ is a variety, but it is neither affine nor projective.

- *V is separated.* Note that V is an open subset of $\mathbb{A}^3_k$ and $\mathbb{A}^3_K$ is separated, hence V is also separated and is actually an open subvariety of $\mathbb{A}^3_k$.
- *V is not affine.* The natural inclusion $\iota : V\hookrightarrow\mathbb{A}^3_k$ induces a homomorphism of rings

  $$\iota^\# : \Gamma(\mathbb{A}^3_k,\mathscr{O}_{\mathbb{A}^3_k})\to\Gamma(V,\mathscr{O}_V)$$

  and it is clear that $\iota^\#$ is nothing else but the natural restriction map. Note that $\mathbb{A}^3_k$ is nonsingular and hence normal and $\dim(V(X_1,X_2)) = 1$. Thus, by the Algebraic Hartogs Theorem, the map $\iota^\#$ is surjective. On the other hand, as $\mathbb{A}^3_k$ is irreducible, the map $\iota^\#$ is also injective. In particular, $\iota^\#$ is an isomorphism of rings. If V is affine, then $\iota^\#$ is an isomorphism if and only if ι is an isomorphism. This is impossible because the inclusion ι is not surjective.
- *V is not projective.* Recall that the image of a complete variety is always closed. In particular, V is closed in $\mathbb{A}^3_k$ which is absurd since $\mathbb{A}^3_k$ is irreducible.

□

(6) (5 points) $(\mathbb{A}^2_k\setminus\{(0,0)\})/\sim$, where $(x_1,x_2)\sim(x_1',x_2')$ if and only if there exists a non-zero number $\lambda\in k^*$ such that $x_1=\lambda x_1'$ and $x_2=\lambda^{-1}x_2'$. (Hint: consider the induced equivalence relation on each open subset $D(X_i)\subset\mathbb{A}^2_k\setminus\{(0,0)\}$ and prove that the quotient of $D(X_i)$ is isomorphic to $\mathbb{A}^1_k$)

Solution. The prevariety V is not separated and hence is not a variety.

- Note that $\mathbb{A}^2_k\setminus\{(0,0)\}=D(X_1)\cup D(X_2)$. Moreover, if $(x_1,x_2)\sim(x_1',x_2')$ and $(x_1,x_2)\in D(X_i)$, then it is obvious that we have $(x_1',x_2')\in D(X_i)$. On each $D(X_i)$, consider the morphism $\pi_i : D(X_i)\to\mathbb{A}^1_k$, $(x_1,x_2)\mapsto x_1x_2$. Then it is easy to check that π induces a bijection $D(X_i)/\sim\to\mathbb{A}^1_k$, $[(x_1,x_2)]\mapsto x_1x_2$.
- According to the argument above, one can easily obtain $V=(\mathbb{A}^1_k\sqcup\mathbb{A}^1_k)/\sim$, where $x\sim y$ if and only if $x=y\neq 0$; that is, V is the line with double origin, which is not separated: consider the natural two inclusion $f_i : \mathbb{A}^1_k\to V$. Then the subset $\{z\in\mathbb{A}^1_k\mid f_1(z)=f_2(z)\}=\mathbb{A}^1_k\setminus\{0\}$ is not closed in $\mathbb{A}^1_k$

□

Problem 2 (40 points)**.** Consider the following Du Val singularity of type A_n $(n\geq 1)$:

$$X_n := V(X_1^2+X_2^2+X_3^{n+1})\subset\mathbb{A}^3_k,$$

where (X_1,X_2,X_3) are the coordinates of $\mathbb{A}^3_k$.

(1) (5 points) Prove that $(0,0,0)$ is the unique singular point of X_n.

Proof. By Krull's Hauptidealsatz, every irreducible component of X_n has dimension 2. Consider the Jacobian matrix of $F=X_1^2+X_2^2+X_3^{n+1}$:

$$J(F)=(2X_1,2X_2,(n+1)X_3^n).$$

Clearly, $J(F)$ has rank $=0$ only at the point $(0,0,0)$, which is the only singular point of X_n. □

(2) (9 points) Determine the blowing-up $\pi_n : \widetilde{X}_n \to X_n$ of X_n at $(0,0,0)$.

Proof. Firstly we recall that the blowing-up of $\mathbb{A}^3_k$ at $(0,0,0)$ is defined as following:

$$\widetilde{\mathbb{A}}^3_k = \{(x_1,x_2,x_3)[y_1 : y_2 : y_3] \in \mathbb{A}^3_k \times \mathbb{P}^2(k) \,|\, x_i y_j = x_j y_i, 1 \le i,j \le 3\}.$$

The local affine descriptions of $\widetilde{\mathbb{A}}^3_k$ are given as following.

- The open subset $U_1 := \widetilde{A}^3_k \cap (\mathbb{A}^3_k \times D(Y_1))$ is isomorphic to $\mathbb{A}^3_k$ given as following:
$$(z_1 : z_2 : z_3) \mapsto (z_1 : z_1 z_2 : z_1 z_3)[1 : z_2 : z_3]$$
- The open subset $U_2 := \widetilde{A}^3_k \cap (\mathbb{A}^3_k \times D(Y_2))$ is isomorphic to $\mathbb{A}^3_k$ given as following:
$$(t_1 : t_2 : t_3) \mapsto (t_1 t_2 : t_2 : t_2 t_3)[t_1 : 1 : t_3]$$
- The open subset $U_3 := \widetilde{A}^3_k \cap (\mathbb{A}^3_k \times D(Y_3))$ is isomorphic to $\mathbb{A}^3_k$ given as following:
$$(w_1 : w_2 : w_3) \mapsto (w_1 w_3 : w_2 w_3 : w_3)[w_1 : w_2 : 1]$$

Denote by $\pi : \widetilde{\mathbb{A}}^3_k \to \mathbb{A}^3_k$ the first projection. By definition, the blowing-up of X_n at $(0,0,0)$ is the strict transform of X_n in $\widetilde{A}^3_k$. The local affine description of $\widetilde{X}_n$ are given as following:

- Over U_1, one can easily obtain the composition
$$F \circ \pi = Z_1^2 + Z_1^2 Z_2^2 + Z_1^{n+1} Z_3^{n+1} = Z_1^2(1 + Z_2^2 + Z_1^{n-1} Z_3^{n+1}).$$
In particular, $\widetilde{X}_n \cap U_1$ is defined by the equation
$$1 + Z_2^2 + Z_1^{n-1} Z_3^{n+1} = 0.$$
- Over U_2, similarly as above, $\widetilde{X}_n \cap U_2$ is defined by the equation
$$T_1^2 + 1 + T_2^{n-1} T_3^{n+1} = 0.$$
- Over U_3, similarly as above, $\widetilde{X}_n \cap U_2$ is defined by the equation
$$W_1^2 + W_2^2 + W_3^{n-1} = 0.$$

□

(3) (5 points) Prove that $\pi_1^{-1}(0)$ is irreducible.

Proof. Denote by $E_n = \pi_n^{-1}(0)$. On the other hand, note that we have

$$E_n \subset \pi^{-1}(0) = \{(0,0,0)\} \times \mathbb{P}^2(k) = \mathbb{P}^2(k).$$

In the following, we aim to find the equation for E_1 in $\mathbb{P}^2(k)$. Firstly we work over the affine open subset U_1. Note that we have $E_n \cap U_1 = \widetilde{X}^3_n \cap \{Z_1 = 0\}$. In particular, $E_1 \cap U_1$ is defined by the equations

$$Z_1 = 1 + Z_2^2 + Z_3^2 = 0.$$

It follows that $E_1 \subset \pi^{-1}(0) = \mathbb{P}^2(k)$ is defined by the equation $1 + Y_2^2 + Y_3^2 = 0$ in the affine open subset $D(Y_1)$. Similarly, $E_1 \cap D(Y_2)$ is defined by the equation $Y_1^2 + 1 + Y_3^2 = 0$ and $E_1 \cap D(Y_3)$ is defined by the equation $Y_1^2 + Y_2^2 + 1 = 0$. Therefore, it is obvious that $E_1 \subset \mathbb{P}^2(k)$ is defined by the equation

$$Y_1^2 + Y_2^2 + Y_3^2 = 0.$$

On the other hand, we can show that E_1 is actually isomorphic to $\mathbb{P}^1$ and hence is irreducible. To see this, consider the 2-nd Veronese embedding $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, which sends the point $[x_0 : x_1]$ to $[x_0^2 : x_0 x_1 : x_1^2]$. Then the image $\nu_2(\mathbb{P}^1)$ is defined as by the equation $Y_1^2 = Y_0 Y_2$. Then the isomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ defined by

$$[y_0 : y_1 : y_2] \mapsto [y_0 + iy_2 : iy_1 : y_0 - iy_2 :]$$

induces an isomorphism between $E_1 \subset \mathbb{P}^2$ and $\nu_2(\mathbb{P}^1) \subset \mathbb{P}^2$. □

(4) (5 points) For $n \ge 2$, prove that $\pi_n^{-1}(0)$ has two irreducible components.

Proof. Similar to (3), for $n \geq 2$, the intersection $E_n \cap D(Y_1)$ is defined by the equation $1 + Y_2^2 = 0$. The intersection $E_n \cap D(Y_2)$ is defined by the equation $Y_1^2 + 1 = 0$ and the intersection $E_n \cap D(Y_3)$ is defined by the equation $Y_1^2 + Y_2^2 = 0$. In particular, one can easily see that $E_1 \subset \mathbb{P}^2(k)$ is defined by the equation $Y_1^2 + Y_2^2 = 0$. On the other hand, note that we have

$$Y_1^2 + Y_2^2 = (Y_1 + iY_2)(Y_1 - iY_2).$$

Moreover, since $E_n' := V(Y_1 + iY_2)$ and $E_n'' := V(Y_1 - iY_2)$ are projective lines in $\mathbb{P}^2(k)$, they are irreducible. Hence, E_n has exactly two irreducible components: E_n' and E_n''. □

(5) (7 points) Prove that $\widetilde{X}_n$ is nonsingular if and only if $n \leq 2$.

Proof.
- For $n = 1$, then $\widetilde{X}_1 \cap U_1$ is defined by the equation $F_1 := 1 + Z_2^2 + Z_3^2 = 0$. Consider the Jacobian matrix of F_1:

$$J(F_1) = (0, 2Z_2, 2Z_3).$$

  By Krull's Hauptidealsatz, $\widetilde{X}_1 \cap U_1$ has dimension 2 and is singular only at the points (z_1, z_2, z_3) with $z_2 = z_3 = 0$. However, one can easily see that the points $(z_1, 0, 0)$ do not lie on $\widetilde{X}_1 \cap U_1$. Hence, $\widetilde{X}_1 \cap U_1$ is irreducible. The same arguments show that both $\widetilde{X}_1 \cap U_2$ and $\widetilde{X}_1 \cap U_3$ are nonsingular and hence $\widetilde{X}_1$ is nonsingular.
- For $n \geq 2$. Then $\widetilde{X}_n \cap U_1$ is defined by the equation $F_1 := 1 + Z_2^2 + Z_1^{n-1} Z_3^{n+1} = 0$. Consider the Jacobian matrix of F_1:

$$J(F_1) = ((n-1)Z_1^{n-1} Z_3^{n+1}, 2Z_2, (n+1)Z_1^{n-1} Z_3^n).$$

  Let $(z_1, z_2, z_3) \in \widetilde{X}_n \cap U_1$ be a point such that $J(F_1)(z_1, z_2, z_3)$ has rank 0. Then clearly we have $z_2 = 0$. In particular, as $F_1(z_1, z_2, z_3) = 0$, it follows that $z_1, z_3 \neq 0$ and consequently $J(F_1)$ has rank 1 at (z_1, z_2, z_3). Hence, $\widetilde{X}_n \cap U_1$ is nonsingular. The same argument applying to $\widetilde{X}_n \cap U_2$ shows that $\widetilde{X}_n \cap U_2$ is nonsingular.
  Finally, the $\widetilde{X}_n \cap U_3$ is defined by the equation $F_3 = W_1^2 + W_2^2 + W_3^{n-1} = 0$. Consider the Jacobian matrix:

$$J(F_3) = (2W_1, 2W_2, (n-1)W_3^{n-2}).$$

  It follows that $\widetilde{X}_3 \cap U_3$ is nonsingular outside $(0, 0, 0)$ and is nonsingular at $(0, 0, 0)$ if and only if $n = 2$.

□

(6) (2 points) For $n \geq 3$, prove that $\widetilde{X}_n$ has a unique singular point p, which is a Du Val singularity of type A_{n-2}; that is, locally $\widetilde{X}_n$ is isomorphic to X_{n-2} at p.

Proof. According to the proof of (5), it is known that for $n \geq 3$, the variety $\widetilde{X}_n$ has only one singular point $(0, 0, 0) \in \widetilde{X}_n \cap U_3$ which is defined by the equation

$$W_1^2 + W_2^2 + W_3^{n-1} = 0.$$

In particular, by definition, this is a Du Val singular point of type A_{n-2}. □

(7) (5 points) Let $Y := V(X_1^2 - X_2 X_3 (X_2 + X_3)) \subset \mathbb{A}_k^3$. Show that the blowing-up $\pi : \widetilde{Y} \to Y$ of Y at $(0, 0, 0)$ has exactly 3 singular points.

Proof. Following the same argument as in (2), one can easily derive the following local affine description of $\widetilde{Y}$:
- Over U_1, the affine variety $\widetilde{Y} \cap U_1$ is defined the following equation

$$G_1 := 1 - Z_1 Z_2^2 Z_3 - Z_1 Z_2 Z_3^2 = 0.$$

  The Jacobian matrix of G_1 is given as

$$J(G_1) = (-Z_2^2 Z_3 - Z_2 Z_3^2, 2Z_1 Z_2 Z_3 - Z_1 Z_3^2, -Z_1 Z_2^2 - 2Z_1 Z_2 Z_3)$$

Let $(z_1, z_2, z_3) \in \widetilde{Y} \cap U_1$ be a point. Then clearly all z_i are non-zero. In particular, if $J(G_1)(z_1, z_2, z_3)$ has rank 0 only if

$$z_2 = -z_3, \qquad 2z_2 = z_3, \qquad z_2 = -2z_3.$$

This implies that $z_2 = z_3 = 0$ which is impossible. Hence $\widetilde{Y} \cap U_1$ is nonsingular.

- Over U_2, the variety $\widetilde{Y} \cap U_2$ is defined by the equation

$$G_2 := T_1^2 - T_2T_3 - T_2T_3^2 = 0.$$

The Jacobian matrix of G_2 is given as

$$J(G_2) = (2T_1, -T_3 - T_3^2, -T_2 - 2T_2T_3).$$

Then $J(G_1)$ has rank 0 at the point (t_1, t_2, t_3) if and only if $(t_1, t_2, t_3) = (0, 0, 0)$ or $(0, 0, -1)$. Clearly that these two points are contained in $\widetilde{Y} \cap U_2$ and they correspond to the points

$$(0, 0, 0)[0 : 1 : 0] \qquad \text{and} \qquad (0, 0, 0)[0 : 1 : -1],$$

in $\mathbb{A}_k^3 \times \mathbb{P}^2(k)$, respectively.

- Over U_3, the variety $\widetilde{Y} \cap U_3$ is defined by the equation

$$G_3 := W_1^2 - W_2^2W_3 - W_2W_3 = 0.$$

The Jacobian matrix of G_2 is given as

$$J(G_2) = (2W_1, -2W_2W_3 - W_3, -W_2^2 - W_2).$$

Then $J(G_2)$ has rank 0 at the point (w_1, w_2, w_3) if and only if $(w_1, w_2, w_3) = (0, 0, 0)$ or $(0, -1, 0)$. Clearly that these two points are contained in $\widetilde{Y} \cap U_3$ and they correspond to the points

$$(0, 0, 0)[0 : 0 : 1] \qquad \text{and} \qquad (0, 0, 0)[0 : -1 : 1],$$

in $\mathbb{A}_k^3 \times \mathbb{P}^2(k)$, respectively.

In conclusion, the singular points of $\widetilde{Y}_n$ are the following three points

$$(0, 0, 0)[0 : 1 : 0], (0, 0, 0)[0 : 1 : -1] = (0, 0, 0)[0 : -1 : 1] \quad \text{and} \quad (0, 0, 0)[0 : 0 : 1]$$

□

(8) (2 points) Show that Y is isomorphic to the Du Val singularity of type D_4 which is defined by the following equation in $\mathbb{A}_k^3$: $X_1^2 + X_2^2X_3 + X_3^3 = 0$.

Proof. Consider the following coordinates changes $f : \mathbb{A}_k^3 \to \mathbb{A}_k^3$

$$(y_1 : y_2 : y_3) \mapsto \left(\frac{iy_1}{2}, \frac{iy_2 + y_3}{2}, \frac{-iy_2 + y_3}{2} \right)$$

which is given in the form of matrix as following:

$$A = \begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & \frac{i}{2} & \frac{1}{2} \\ 0 & \frac{-i}{2} & \frac{1}{2} \end{pmatrix}$$

It is easy to see that $\det(A) \neq 0$ and hence f is an isomorphism. Moreover, a straightforward computation shows that $f^{-1}(Y)$ is defined by the equation

$$Y_1^2 + Y_2^2Y_3 + Y_3^3 = 0,$$

which is exactly the Du Val singularity of type D_4. □

Problem 3 (30 points)**.** For two integers $n \geq 1$, $d \geq 2$. Let $X_{n,d} \subset \mathbb{P}^{n+1}(k)$ be the n-dimensional Fermat hypersurface of degree $d \geq 2$ defined by the following equation

$$X_0^d + X_1^d + \cdots + X_n^d + X_{n+1}^d = 0,$$

where $[X_0 : X_1 : \cdots : X_n : X_{n+1}]$ is the homogeneous coordinates of $\mathbb{P}^{n+1}(k)$.

(1) (5 points) Prove that $X_{n,d}$ is nonsingular.

Proof. It is enough to prove it for $X_{n,d} \cap D(X_i)$. Without loss of generality, we may assume that $i = 0$, then $X_{n,d} \cap D(X_0) \subset D(X_0) = \mathbb{A}_k^{n+1}$ is defined by the equation

$$F_0 := 1 + X_1^d + \cdots + X_{n+1}^d = 0,$$

where $(X_1, \cdots, X_{n+1})$ are the coordinates of $D(X_0) = \mathbb{A}_k^{n+1}$. The the Jacobian matrix of F_1 is given as following

$$J(F_0) = (dX_1^{d-1}, \cdots, dX_{n+1}^{d-1}).$$

One can easily derive that $J(F_0)$ has constant rank 1 along $X_{n,d} \cap D(X_0)$ and hence $X_{n,d} \cap D(X_0)$ is nonsingular. Similarly, one can see that $X_{n,d} \cap D(X_i)$ is nonsingular for any $0 \leq i \leq n+1$ and hence $X_{n,d}$ is nonsingular. □

(2) (5 points) Let $H \subset \mathbb{P}^{n+1}(k)$ be the hyperplane defined by the equation $X_0 = 0$. Prove that $X_{n,d}$ is linearly equivalent to dH as divisors in $\mathbb{P}^{n+1}(k)$.

Proof. Consider the non-zero rational function $\phi := \frac{X_0^d + \cdots + X_{n+1}^d}{X_0^d} \in K(\mathbb{P}^{n+1})^*$. Then for each $0 \leq i \leq n+1$, it is clear that we have

$$\operatorname{div}(\phi)|_{D_{X_i}} = X_{n,d}|_{D(X_i)} - dH|_{D(X_i)}.$$

Hence, by definition $X_{n,d}$ is linearly equivalent to dH. □

(3) (5 points) Let $\iota : X_{n,d} \hookrightarrow \mathbb{P}^{n+1}(k)$ be the natural closed immersion. Show that there exists a natural short exact sequence as following

$$0 \to \mathscr{O}_{\mathbb{P}^{n+1}(k)}(-d) \to \mathscr{O}_{\mathbb{P}^{n+1}(k)} \to \iota_* \mathscr{O}_{X_{n,d}} \to 0.$$

Proof. Denote by $\mathscr{I}$ the ideal sheaf of $X_{n,d}$ in $\mathbb{P}^{n+1}$. Then we have the following exact sequence

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}^{n+1}} \to \iota_* \mathscr{O}_{X_{n,d}} \to 0.$$

It is enough to show that $\mathscr{I}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^{n+1}}(-d)$. Note that $X_{n,d}$ is a prime divisor in $\mathbb{P}^{n+1}$, we have $\mathscr{I} \cong \mathscr{O}_{\mathbb{P}^{n+1}}(-X_{n,d})$. To see that $\mathscr{O}_{\mathbb{P}^{n+1}}(-X_{n,d}) \cong \mathscr{O}_{\mathbb{P}^{n+1}}(-d)$, we observe that $\mathscr{O}_{\mathbb{P}^{n+1}}(X_{n,d})$ is isomorphic to $\mathscr{O}_{\mathbb{P}^{n+1}}(d)$ since $F = X_0^d + \cdots + X_{n+1}^d$ is a non-zero element of $\Gamma(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(d))$ and $X_{n,d} = \operatorname{div}(F)$. □

(4) (5 points) Deduce that $H^q(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) = 0$ for $1 \leq q \leq n-1$.

Proof. Since $X_{n,d}$ is a closed subvariety of $\mathbb{P}^{n+1}$, we have the following equality

$$H^q(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) = H^q(\mathbb{P}^{n+1}, \iota_*(\mathscr{O}_{X_{n,d}}(m))).$$

On the other hand, since $\mathscr{O}_{X_{n,d}}(m) \cong \iota^* \mathscr{O}_{\mathbb{P}^{n+1}}(m)$ by definition and $\mathscr{O}_{\mathbb{P}^{n+1}(m)}$ is locally free, according to the projection formula, we have

$$\iota_*(\mathscr{O}_{X_{n,d}}(m)) = \iota_* \iota^* \mathscr{O}_{\mathbb{P}^{n+1}}(m) = \iota_* \mathscr{O}_{X_{n,d}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(m).$$

Since $\mathscr{O}_{\mathbb{P}^{n+1}}(m)$ is locally free, tensoring $\mathscr{O}_{\mathbb{P}^{n+1}}(m)$ with the short exact sequence in (3) yields a short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^{n+1}}(m-d) \to \mathscr{O}_{\mathbb{P}^{n+1}}(m) \to \iota_* \mathscr{O}_{X_{n,d}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(m) \to 0.$$

For $1 \leq q \leq n-1$, the short exact sequence above yields an exact sequence of vector spaces

$$H^q(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(m)) \to H^q(\mathbb{P}^{n+1}, \iota_* \mathscr{O}_{X_{n,d}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(m)) \to H^{q+1}(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(m-d))$$

Recall that we have $H^i(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(j)) = 0$ for $1 \leq i \leq n$ and $j \in \mathbb{Z}$ from the course or using Kodaira's Vanishing Theorem + Serre Duality. In particular, this implies that for $1 \leq q \leq n-1$ and any $m \in \mathbb{Z}$, we have

$$H^q(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) = H^q(\mathbb{P}^{n+1}, \iota_*\mathscr{O}_{X_{n,d}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(m)) = 0.$$

This finishes the proof. □

(5) (8 points) Compute $\dim_k H^0(X_{n,d}, \mathscr{O}_{X_{n,d}}(m))$ and $\dim_k H^n(X_{n,d}, \mathscr{O}_{X_{n,d}}(m))$.

Solution. • For $q = 0$, as in (4), we have the following short exact sequence of vector spaces

$$0 \to H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(m-d)) \to H^0(\mathbb{P}^{n+1}, \mathbb{P}^{n+1}(m)) \to H^0(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) \to 0.$$

In particular, this implies that we have

$$\dim_k H^0(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) = \dim_k H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(m)) - \dim_k H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(m-d)).$$

Recall that for any $j \in \mathbb{Z}$, we have $H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(j)) = S_j$, where S_j is the set of homogeneous polynomials of degree j in $k[X_0, \cdots, X_{n+1}]$. In particular, we have

$$\dim_k H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(j)) = \begin{cases} 0 & \text{if} \quad j < 0; \\ \binom{n+j+1}{j} = \binom{n+j+1}{n+1} & \text{if} \quad j \geq 0. \end{cases}$$

As a consequence, we get

$$\dim_k H^0(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) = \begin{cases} 0 & \text{if} \quad m < 0; \\ \binom{n+m+1}{m} & \text{if} \quad 0 \leq m < d; \\ \binom{n+m+1}{m} - \binom{n+m-d+1}{m-d} & \text{if} \quad m \geq d. \end{cases}$$

• For $q = n$, as in (4), we have the following short exact sequence of vector spaces

$$0 \to H^n(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) \to H^{n+1}(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(m-d)) \to H^{n+1}(\mathbb{P}^{n+1}, \mathbb{P}^{n+1}(m)) \to 0.$$

The last term follows from the Grothendieck Vanishing Theorem and the fact that $\dim(X_{n,d}) = n$. On the other hand, by Serre Duality and the fact $\Omega_{\mathbb{P}^{n+1}} \cong \mathscr{O}_{\mathbb{P}^{n+1}}(-n-2)$, for any $j \in \mathbb{Z}$, we have

$$\dim_k H^{n+1}(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(j)) = \dim_k H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(-n-2-j)) = \dim_k S_{-n-2-j}.$$

As a consequence, we obtain

$$\dim_k H^n(X_{n,d}, \mathscr{O}_{\mathbb{P}^{n+1}}(m)) = \begin{cases} 0 & \text{if} \quad m > d-n-2; \\ \binom{d-m-1}{d-n-m-2} = \binom{d-m-1}{n+1} & \text{if} \quad -n-2 \leq m \leq d-n-2; \\ \binom{d-m-1}{n+1} - \binom{-m-1}{n+1} & \text{if} \quad m \leq -n-2. \end{cases}$$

□

(6) (2 points) Prove that if $X_{1,d}$ is isomorphic to $X_{1,d'}$, then $d = d'$.

Proof. By (5), taking $m = 0$, we have

$$\dim_k H^1(X_{1,d}, \mathscr{O}_{X_{1,d}}) = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.$$

In particular, if $X_{1,d}$ is isomorphic to $X_{1,d'}$ is isomorphic to $X_{1,d'}$, then we must have

$$\dim_k H^1(X_{1,d}, \mathscr{O}_{X_{1,d}}) = \dim_k H^1(X_{1,d'}, \mathscr{O}_{X_{1,d'}}).$$

This implies immediately that $d = d'$. □

Remark. In the proof above, *a priori* we can not apply the argument to other intergers $m \neq 0$ because $\mathscr{O}_{X_{1,d}}(1)$ depends on the embedding of $X_{1,d}$ into projective spaces, which is not canonical. Thus, if we want to apply the same argument to other $m \neq 0$, we do need to show that any isomorphism $f : X_{1,d} \to X_{1,d'}$ induces also an isomorphism $f^*\mathscr{O}_{X_{1,d'}}(m) \cong \mathscr{O}_{X_{1,d'}}(m)$. □