

Exercise sheet 4

Exercise 1 (Flex points and Hessian). Let k be an algebraically closed field of characteristic $p \geq 0$, and let F be a reduced projective plane curve of degree d . Let H_F be the Hessian curve of F , i.e. the curve defined by the determinant of the Hessian matrix of F .

- (1) Prove that we always have

$$z^2 \cdot H_F = \det \begin{pmatrix} d(d-1)F & (d-1)F_x & (d-1)F_y \\ (d-1)F_x & F_{xx} & F_{xy} \\ (d-1)F_y & F_{xy} & F_{yy} \end{pmatrix}.$$

Here the right-hand side is obtained by eliminating the derivatives with respect to z using the Euler identity.

Hint: Use the Euler identity for homogeneous polynomials.

- (2) Prove that $H_F(P) = 0$ if P is a singular point of F . Moreover, show that if p divides $d-1$, then $H_F \equiv 0$.
- (3) From now on assume that $d \geq 3$ and either $p = 0$ or $p > d$.

- (a) Let $P \in F$ be a regular point, and let G be the tangent line to F at P . Assume that G is not a component of F . Show that, after a suitable choice of affine coordinates, we may assume that $P = (0,0)$ and $G = y$. Let $f(x,y) = 0$ be the corresponding affine equation of F . Show that the intersection multiplicity $\mu_P(F, G)$ is the order of y in the local ring $\mathcal{O}_{F,P}$.

- (b) Assume that the local equation of F has the form

$$f(x,y) = y\psi(x,y) + x^\mu\phi(x),$$

where $\phi(0) \neq 0$ and $\psi(0,0) \neq 0$. Show that the image of x generates the maximal ideal of $\mathcal{O}_{F,P}$ and that $v_P(y) = \mu$.

- (c) Consider the determinant

$$\Delta = \det \begin{pmatrix} d(d-1)f & (d-1)f_x & (d-1)f_y \\ (d-1)f_x & f_{xx} & f_{xy} \\ (d-1)f_y & f_{xy} & f_{yy} \end{pmatrix}.$$

Show that the image of Δ in the local ring $\mathcal{O}_{F,P}$ has valuation $v_P(\Delta) = \mu - 2$. Deduce that $\mu_P(F, H_F) = \mu_P(F, G) - 2$.

- (d) Prove that for every regular point $P \in F$ whose tangent line is not a component of F , one has $\mu_P(F, G) = \mu_P(F, H_F) + 2$.
- (e) Assume that $H_F \equiv 0 \pmod{F}$. Show that for every regular point $P \in F$, $P \in F \cap H_F$ if and only if P is a flex point. Deduce that the intersection of F with its Hessian curve consists precisely of the singular points of F and the flex points of F .
- (f) Assume that $F = zJ$, where J is a product of linear homogeneous polynomials. Show that $H_F \equiv 0 \pmod{F}$.

Hint: You may use the relation

$$Z^2 H_F \equiv \left(\frac{d-1}{d-2}\right)^2 z^5 H_J \pmod{F}$$

and argue by induction on the degree.

- (g) Deduce that $H_F \equiv 0 \pmod{F}$ if and only if F is a union of lines.

Exercise 2 ($l(D)$ for elliptic curves). Let E be an elliptic curve, and fix a point $O \in E$ as the origin of the group law on E . Let D be a divisor on E . Show that $l(D)$ is given by the following rules:

- (1) If $\deg D < 0$, then $l(D) = 0$.
- (2) If $\deg D > 0$, then $l(D) = \deg D$.
- (3) If $\deg D = 0$, we can write $D = P_1 + \cdots + P_n - Q_1 - \cdots - Q_n$ for some $n \geq 0$ and points $P_1, \dots, P_n, Q_1, \dots, Q_n \in E$. Then

$$l(D) = \begin{cases} 1, & \text{if } P_1 + \cdots + P_n = Q_1 + \cdots + Q_n, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 3. Let E be the elliptic curve $y^2z - x^3 - \lambda xz^2 - \mu z^3$ with $4\lambda^3 + 27\mu^2 \neq 0$. Fix the point $O = [0 : 1 : 0]$ as the base point of the group law on E .

- (1) Compute the coordinates of $P \oplus Q$ and $\ominus P$ for arbitrary points $P, Q \in E$.
- (2) Show that the subgroup $\{D \in \text{Pic}(F) \mid 2D \sim 0\}$ of $\text{Pic}(F)$ has exactly 4-elements and isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Hint: Translate the problem to the group structure on E .

- (3) Assume now that $\mu = 0$ and E is defined over a field of characteristic $p > 0$ (so that $\mathbb{Z}/p\mathbb{Z}$ is a subfield of k) and with $\lambda \in \mathbb{Z}/p\mathbb{Z}$. Assume moreover that $\lambda \neq 0$. Show that:
 - (a) If $p \equiv 3 \pmod{4}$, then $\#(V(E) \cap \mathbb{P}_{\mathbb{Z}/p\mathbb{Z}}^2) = p + 1$.
 - (b) If $p \equiv 1 \pmod{4}$, then $\#(V(E) \cap \mathbb{P}_{\mathbb{Z}/p\mathbb{Z}}^2)$ is even.

Exercise 4. Let E be an elliptic curve corresponding to a torus $\mathbb{C}/\Lambda$. Show that the group structure on E induced by $\text{Pic}^0(F)$ using the point $[0 : 0 : 1]$ as the base point is the same as the one induced by the natural group structure of $\mathbb{C}/\Lambda$.