Exercise sheet 3

Exercise 1 (Rational functions and multiplicities)**.** Let $F \subset \mathbb{P}^2$ be a smooth irreducible projective curve. Let $F^i \subset \mathbb{A}^2$ be an affine part of F, and let $P \in F^i$ be a point. Let $S(F)$ be the homogeneous coordinate ring of F, and let $A(F^i)$ be the coordinate ring of F^i.

(1) Prove that $K(F) \cong K(F^i)$.

(2) Prove that $\mathcal{O}_{F^i,P} \cong \mathcal{O}_{\mathbb{A}^2,P}/(F^i)$. Hence prove that for any nonzero $\psi \in A(F^i)$,

$$\mu_P(\psi) = \dim_k \mathcal{O}_{F^i,P}/\langle h \rangle.$$

(3) Prove that there exists a rational function $\psi \in K(F^i)$ such that $\mu_P(\psi) = -1$, and $\mu_Q(\psi) \geq 0$ for every $Q \neq P$.

(4) Prove that $S(F) = \bigoplus_{d \geq 0} S_d(F)$.

(5) For a nonzero rational function $\varphi = \frac{f}{g} \in K(F)$, prove that $\mu_P(\varphi)$ is well-defined. Moreover, if $P \in F^i$, prove that $\mu_P(\varphi) = \mu_P(\varphi^i)$, where

$$\varphi^i = \frac{f^i}{g^i}.$$

(6) Assume $F = y^2 z + x^3 - xz^2$. Let $P = [0:0:1] \in F$ and

$$\varphi = \frac{x^2}{y^2 + yz}.$$

   (a) Compute $m = \mu_P(\varphi)$.
   (b) Find a local coordinate $t \in \mathcal{O}_{F,P}$.
   (c) Find an expression $\varphi = ct^m$, where $c = \frac{f}{g} \in \mathcal{O}_{F,P}$ is a unit and f, g are homogeneous polynomials of the same degree with

$$f(P) \neq 0, \qquad g(P) \neq 0.$$

   (d) Compute $\operatorname{div}(\varphi)$.

Exercise 2 (Divisors)**.** Let F be a smooth irreducible projective curve.

(1) Let D be a divisor on F. Prove that $\deg(D) = 0$ and $l(D) > 0$ if and only if $D \sim 0$.

(2) Let f be a homogeneous polynomial with $\operatorname{div}(f) = D + E$ for two divisors D and E on F. Prove that if $D \sim D$ and $D' + E$ is effective, then there exists a homogeneous polynomial g of degree d such that $\operatorname{div}(g) = D' + E$.

(3) Assume that F is a smooth plane cubic. Let P, Q, R, S be four distinct points on F such that $P + Q \sim R + S$. Assume that the two lines $\overline{PQ}$ and $\overline{RS}$ are distinct. Show that

$$\overline{PQ} \cap \overline{RS} \in F.$$

Exercise 3 (Riemann–Roch Theorem)**.** Let F be a smooth irreducible projective curve of degree d and genus g. Denote by $k[x,y,z]_n$ the space of homogeneous polynomials of degree n.

(1) Assume that the line $z = 0$ is not a component of F. For any $n \geq d$, let $D = \operatorname{div}(z^n)$.

   (a) Prove that there is an exact sequence

$$0 \to k[x,y,z]_{n-d} \xrightarrow{\times F} k[x,y,z]_n \xrightarrow{/z^n} L(D) \to 0.$$

   (b) Prove that $l(D) = \deg(D) + 1 - g$.

(2) **(Brill–Noether reciprocity)** Suppose $D + D' \sim K_F$. Prove that
$$l(D) - l(D') = \frac{1}{2}(\deg D - \deg D').$$
(3) Let D be a divisor such that $\deg(D) = 2g - 2$ and $l(D) = g$. Prove that $D \sim K_F$.
(4) Let $D, D' \geq 0$ be effective divisors. Assume that there exists $0 \neq \psi \in L(D)$ such that $\psi \notin L(D - E)$ for every nonzero effective divisor $E \leq D'$. Prove that multiplication by ψ induces an isomorphism
$$L(D')/L(0) \longrightarrow L(D + D')/L(D).$$
(5) **(Clifford's theorem)** Let D be a divisor such that $l(D) > 0$ and $l(K_F - D) > 0$. Prove that
$$l(D) \leq \frac{1}{2}\deg(D) + 1.$$