Exercise sheet 2

Exercise 1 (Homogeneous polynomials)**.** Denote by $\mathbb{P}^n$ the projective n-space over a fixed field k.

(1) Let $f \in k[x_0, \ldots, x_n]$ be a polynomial such that it defines a well-defined function on $\mathbb{P}^n$, i.e.
$$f(\lambda x_0, \ldots, \lambda x_n) = f(x_0, \ldots, x_n)$$
for every $\lambda \in k^\times$. Prove that f must be a constant if k is infinite. Give an example showing that this is not true if k is finite.

(2) Prove that an irreducible factor of a homogeneous polynomial is again homogeneous.

(3) Prove the Euler identity: For any homogeneous polynomial
$$F \in k[x_0, \ldots, x_n]$$
of degree d, we have
$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = dF.$$

(4) By a *projective coordinate transformation*, we mean a map $f : \mathbb{P}^n \to \mathbb{P}^n$ of the form
$$[x_0 : \cdots : x_n] \longmapsto [f_0(x_0, \ldots, x_n) : \cdots : f_n(x_0, \ldots, x_n)],$$
where $f_0, \ldots, f_n \in k[x_0, \ldots, x_n]$ are linearly independent homogeneous linear polynomials. Let $P_1, \ldots, P_{n+2} \in \mathbb{P}^n$ be points such that any $n+1$ representatives of these points in k^{n+1} are linearly independent. Similarly, let $Q_1, \ldots, Q_{n+2} \in \mathbb{P}^n$ satisfy the same condition. Show that there exists a projective coordinate transformation f such that $f(P_i) = Q_i$ for all $1 \leq i \leq n+2$.

(5) Assume that k is algebraically closed. Prove that every homogeneous polynomial in two variables is a product of linear polynomials.

(6) Let F and G be two smooth real projective conics with non-empty sets of real points. Show that there exists a projective coordinate transformation of $\mathbb{P}^2$ that sends F to G.

Exercise 2 (Intersection multiplicities)**.** Let $F, G \subset \mathbb{P}^2$ be two projective curves, and let F^i, G^i denote their restrictions to a fixed affine chart. Let $P \in F \cap G$.

(1) Assume that $P \in F^i \cap G^i$. Prove that $\mu_P(F, G) = \mu_P(F^i, G^i)$.

(2) Prove that the multiplicity $m_P(F)$ and the tangent cone of F at P are independent of the choice of affine coordinates on $\mathbb{P}^2$.

(3) Let F be an irreducible projective cubic curve such that $P = [0 : 0 : 1] \in F$ is a cusp and the tangent line at P is $x_1 = 0$. Prove that
$$F = ax_1^2 x_2 - bx_0^3 - cx_0^2 x_1 - dx_0 x_1^2 - ex_1^3.$$
Find projective coordinate transformations which make
   (a) $a = b = 1$;
   (b) $c = 0$;
   (c) $d = e = 0$.
Conclude that every irreducible cubic with a cusp is projectively equivalent to $x_1^2 x_2 = x_0^3$. Show that this curve has no other singularities.

(4) Assume that k is algebraically closed. Prove that an irreducible cubic curve is either smooth or has exactly one singular point, which is a node or a cusp.

Exercise 3 (Hessian)**.** Let $F(x_0, x_1, x_2)$ be a homogeneous polynomial of degree $d \geq 3$ defining a projective plane curve in $\mathbb{P}^2$ with homogeneous coordinates $[x_0 : x_1 : x_2]$. The *Hessian matrix* of F is

$$\mathrm{Hess}(F) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{0\leq i,j\leq 2},$$

and the *Hessian curve* of F is defined by $H_F = \det(\mathrm{Hess}(F))$.

(1) Show that the Hessian is compatible with projective coordinate transformations. More precisely, if a projective coordinate transformation takes F to F', then it takes the Hessian curve of F to the Hessian curve of F' up to multiplication by a nonzero scalar.

(2) Let $P \in F$ be a smooth point, and assume that the characteristic of k is zero. Show that $H_F(P) = 0$ if and only if $\mu_P(F, T_P F) \geq 3$. Such a point is called an *inflection point* of F.

Exercise 4 (Applications of Bézout's Theorem and Max Noether's Theorem)**.** Assume that k is algebraically closed.

(1) Show that every smooth projective curve is irreducible. Is the same statement true for affine curves?

(2) **(Cayley–Bacharach)** Let F and G be smooth projective cubics such that $F \cap G = \{P_1, \dots, P_9\}$ consists of nine distinct points. Let E be another cubic containing the first eight points $P_1, \dots, P_8$. Prove that E also contains P_9.

Hint: Apply Max Noether's Fundamental Theorem to a suitable curve.

(3) Let F and G be projective cubic curves with no common component such that $F \cap G = \{P_1, \dots, P_9\}$ consists of nine distinct points. Let H be a projective conic such that $H \cap F = \{P_1, \dots, P_6\}$ consists of six distinct points. Assume that F is smooth at these six points. Prove that P_7, P_8, P_9 lie on a line.

(4) **(Pascal's Theorem)** Let F be an irreducible projective conic. Pick six distinct points $P_1, \dots, P_6$ on F. Show that the three intersection points

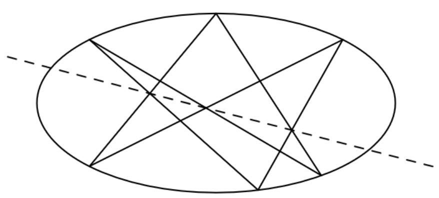
$$P = \overline{P_1P_2} \cap \overline{P_4P_5}, Q = \overline{P_2P_3} \cap \overline{P_5P_6}, R = \overline{P_3P_4} \cap \overline{P_6P_1}$$

are collinear.

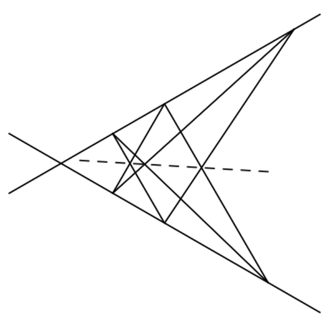
(5) **(Pappus)** Let L_1, L_2 be two lines. Let $P_1, P_2, P_3 \in L_1$, $Q_1, Q_2, Q_3 \in L_2$ be distinct points and assume that none of them is the intersection point $L_1 \cap L_2$. Let L_{ij} be the line joining P_i and Q_j. Define

$$R_1 = L_{23} \cap L_{32}, \qquad R_2 = L_{31} \cap L_{13}, \qquad R_3 = L_{12} \cap L_{21}.$$

Prove that R_1, R_2, R_3 are collinear.



Pascal's Theorem



Pappus' Theorem

(6) **(Converse of Pascal's Theorem)** Let $P_1, \dots, P_6 \in \mathbb{P}^2$ be distinct points such that the six lines $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_6P_1}$ are distinct. Let P, Q, R be the intersections of opposite sides as above. Assume that P, Q, R are collinear. Prove that the six points $P_1, \dots, P_6$ lie on a conic.