Exercise sheet 1

Exercise 1 (Affine varieties)**.** Determine whether the following subsets of $\mathbb{A}^n$ are affine varieties (as defined in the course).

(1) $\{(t, t^2, t^3) \in \mathbb{A}^3_k \mid t \in k\}$
(2) $\{(\cos(t), \sin(t)) \in \mathbb{A}^2_{\mathbb{R}} \mid t \in \mathbb{R}\}$
(3) the set of points in $\mathbb{A}^2_{\mathbb{R}}$ whose polar coordinates (r, θ) satisfy the equation $r = \sin(\theta)$
(4) $\{(x, y) \in \mathbb{A}^2_{\mathbb{R}} \mid y = \sin(x)\}$
(5) $\{(x, y) \in \mathbb{A}^2_{\mathbb{C}} \mid |x|^2 + |y|^2 = 1\}$
(6) $\{(\cos(t), \sin(t), t) \in \mathbb{A}^3_{\mathbb{R}} \mid t \in \mathbb{R}\}$

Exercise 2 (Real plane algebraic curves)**.** Let $k = \mathbb{R}$.

(1) Prove that every affine variety in $\mathbb{A}^2$ is of the form $V(F)$ for some $F \in k[x, y]$.
(2) Find the irreducible components of the following curves:

$$y^2 - xy - x^2y + x^3, \quad y^2 - x(x^2 - 1), \quad x^3 + x - x^2y - y.$$

(3) Prove that the curve $F = y^2 + x^2(x-1)^2$ is irreducible over $\mathbb{R}$. Find the zero locus $V(F)$.

Exercise 3 (Pythagorean triples)**.** Let k be a field with odd characteristic. Let $F = x^2 + y^2 - 1 \in k[x, y]$ be the equation of a conic in $\mathbb{A}^2$.

(1) Considering the intersection points of an arbitrary line L with slope t through $(-1, 0)$ with F, show that the set of points of F is

$$V(F) \coloneqq \{(-1,0)\} \cup \left\{ \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \} t \in k, t^2 \neq -1 \right\}$$

(2) Using the previous statement, prove that the integer solutions (a, b, c) of the equation $a^2 + b^2 = c^2$ (the so-called *Pythagorean triples*) are, up to exchanging a and b, exactly the triples of the form

$$\lambda(u^2 - v^2, 2uv, u^2 + v^2)$$

with $\lambda, u, v \in \mathbb{Z}$.

Exercise 4 (Intersection multiplicities)**.** Let F and G be two curves passing through the origin. Show:

(1) If F and G have no common component, then the family $(F^n)_{n \in \mathbb{N}}$ is linearly independent in $\mathscr{O}_0/\langle G \rangle$.
(2) If F and G have a common component that passes through the origin, then $\mu_0(F, G) = \infty$.
(3) Let H be another curve passing through the origin. Assume that H is regular at the origin. Prove that $\mu_0(H, F + G) \geq \min\{\mu_0(H, F), \mu_0(H, G)\}$. Give an example to show that this may be false if H is not regular at the origin.

Now we assume that F and G have no common components. Show:

(1) There is a number $n \in \mathbb{N}$ such that $x^n = y^n = 0$ in $\mathscr{O}_0/\langle F, G \rangle$.
(2) Every element of $\mathscr{O}_0/\langle F, G \rangle$ has a polynomial representative.
(3) $\mu_0(F, G) < \infty$.

(4) Assume that G is a line, parametrized by $(a+tb, c+td)$, $t \in k$. Then we define $h(t) = F(a+tb, c+td)$. Factor $h(t) = \lambda \prod (t-\lambda_i)^{e_i}$, λ_i distinct. Show that there is a natural one-to-one correspondence between the λ_i and the points $P_i \in G \cap F$. Show that under this correspondence, we have $\mu_{P_i}(G, F) = e_i$. In particular, we obtain

$$\sum_{P \in G \cap F} \mu_P(G, F) \leq \deg(F).$$

(5) Let $F = (x^2+y^2)^2 + 3x^2y - y^3$ and let $G = (x^2+y^2)^3 - 4x^2y^2$. Calculate $\mu_0(F, G)$.

Exercise 5 (Singular points)**.** Let $k = \mathbb{C}$. Find the multiple points, and the tangents at the multiple points, for each of the following curves:

(1) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$
(2) $x^4 + y^4 - x^2y^2$
(3) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$
(4) $y^2 + (x^2-5)(4x^4 - 20x^2 + 25)$

Exercise 6 (Double points and cusps)**.** Let $P \in F$ be a double point, i.e., $m_P(F) = 2$. Suppose that F has only one tangent L at P.

(1) Show that $\mu_P(F, L) \geq 3$. The curve F is said to have a *cusp* at P if $\mu_P(F, L) = 3$.
(2) Suppose $P = (0,0)$, and $L = y$. Show that P is a cusp if and only if $\frac{\partial^3 F}{\partial x^3} \neq 0$.
(3) Give an example of a real curve with a cusp, and draw a picture of it.
(4) If F has a cusp at P, prove that F has only one irreducible component passing through P.
(5) If F and G have a cusp at P, what is the minimum possible value for the intersection multiplicity $\mu_P(F, G)$?
(6) A point P on a curve F is called a *hypercusp* if $m_P(F) > 1$, F has only one tangent line L at P, and $\mu_P(L, F) = m_P(F) + 1$. Generalize the results above to this case.

Exercise 7 (Flex points)**.** Let $P \in F$ be a regular point. Then P is called *flex* if $\mu_P(F, L) \geq 3$, where L is the tangent to F at P. The flex is called *ordinary* if $\mu_P(F, L) = 3$, a *higher* flex otherwise.

(1) Let $F = y - x^n$. For which n does F have a flex at $P = (0,0)$, and what kind of flex?
(2) Suppose $P = (0,0)$, $L = y$ is the tangent to

$$F = y + ax^2 + \text{higher order terms}$$

at P. Show that P is a flex on F if and only if $a = 0$. Determine when P is a higher flex.