Exercise sheet 2

Exercise 1 (Morphisms of sheaves)**.** Let $\mathscr{F}$ and $\mathscr{E}$ be sheaves of abelian groups on a topological space X.

(1) Let $f\colon \mathscr{F} \to \mathscr{E}$ be a morphism of sheaves. For each $x \in X$, define a map $f_x\colon \mathscr{F}_x \to \mathscr{E}_x$ by
$$f_x([s]) \coloneqq [f_U(s)],$$
where $s \in \mathscr{F}(U)$ and U is an open neighborhood of x. Prove that f_x is a well-defined homomorphism of abelian groups.

(2) Recall that a morphism $f\colon \mathscr{F} \to \mathscr{E}$ is called injective if $f_x\colon \mathscr{F}_x \to \mathscr{E}_x$ is injective for every $x \in X$. Prove that f is injective if and only if $f_U\colon \mathscr{F}(U) \to \mathscr{E}(U)$ is injective for every open subset $U \subseteq X$.

(3) Let $X = \{z \in \mathbb{C} \mid |z| = 1\} = S^1$. For each open subset $U \subseteq X$, define
$$\mathscr{E}(U) \coloneqq \{s\colon U \to \mathbb{C} \mid s \text{ is continuous}\}$$
and
$$\mathscr{Q}(U) \coloneqq \{s\colon U \to \mathbb{C}^* \mid s \text{ is continuous}\}.$$
Define a morphism $g\colon \mathscr{E} \to \mathscr{Q}$ by $g_U\colon \mathscr{E}(U) \to \mathscr{Q}(U)$, $s \mapsto \exp \circ s$. Prove that g is surjective; that is, $g_x\colon \mathscr{E}_x \to \mathscr{Q}_x$ is surjective for every $x \in X$.

(4) In (3), prove that $g_X\colon \mathscr{E}(X) \to \mathscr{Q}(X)$ is not surjective.

Exercise 2 (Cohomology of the projective line)**.** Denote by $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the *extended complex plane*, equipped with the one-point compactification topology.

(1) Prove that $\mathbb{P}^1$ is homeomorphic to $\bar{\mathbb{C}}$.

(2) For every $k \in \mathbb{Z}$, prove that
$$H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k)) \cong \begin{cases} 0 & \text{if } k < 0, \\ \mathbb{C} & \text{if } k = 0, \\ \mathbb{C}[x_0, x_1]_k & \text{if } k \geq 1, \end{cases}$$
where $\mathbb{C}[x_0, x_1]_k$ is the vector space of homogeneous polynomials of degree k in x_0, x_1, including the zero polynomial.

(3) Let $\mathcal{U} = \{U_0, U_1\}$ be the standard open covering of $\mathbb{P}^1$, where
$$U_i = \{[x_0 : x_1] \in \mathbb{P}^1 \mid x_i \neq 0\}, \qquad i = 0, 1.$$
For every $k \in \mathbb{Z}$ and every $q \in \mathbb{Z}_{\geq 0}$, compute the Čech cohomology $\check{H}^q(\mathcal{U}, \mathscr{O}_{\mathbb{P}^1}(k))$.

(4) For every $k \in \mathbb{Z}$ and every $q \in \mathbb{Z}_{\geq 0}$, compute the sheaf cohomology $H^q(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k))$.

(5) Prove that the holomorphic tangent bundle $T_{\mathbb{P}^1}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(2)$.

Exercise 3 (Chern–Weil theorem)**.** Let X be a differential manifold and let $E \to X$ be a complex vector bundle.

(1) Let ∇ and ∇' be two connections on E. Prove that $\nabla - \nabla' \in \mathcal{A}^1_{\mathbb{C}}(X, \mathrm{End}(E))$.

(2) For integers $q, q' \geq 0$, let $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$ and $\beta \in \mathcal{A}^{q'}_{\mathbb{C}}(X, \mathrm{End}(E))$. Define their *supercommutator* by
$$[\alpha, \beta] \coloneqq \alpha \wedge \beta - (-1)^{qq'} \beta \wedge \alpha.$$
Prove that $\mathrm{tr}[\alpha, \beta] = 0$.

(3) For a connection ∇ and $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$, with $q \geq 0$, define

$$[\nabla, \alpha] \colon \mathcal{A}^0_{\mathbb{C}}(X, E) \longrightarrow \mathcal{A}^{q+1}_{\mathbb{C}}(X, E), \quad s \longmapsto \nabla(\alpha \wedge s) - (-1)^q \alpha \wedge \nabla s.$$

Prove that $[\nabla, \alpha] \in \mathcal{A}^{q+1}_{\mathbb{C}}(X, \mathrm{End}(E))$.

(4) (**Bianchi identity**) For every integer $k > 0$, prove that $[\nabla, \Theta^k_\nabla] = 0$.

(5) For every $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$, with $q \geq 0$, prove that $\mathrm{tr}([\nabla, \alpha])$ is independent of the choice of the connection ∇.

(6) For every $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$, with $q \geq 0$, prove that $\mathrm{d}(\mathrm{tr}\,\alpha) = \mathrm{tr}[\nabla, \alpha]$.

(7) Let $f(t) = a_0 + a_1 t + \cdots + a_m t^m \in \mathbb{C}[t]$. Define

$$f(\Theta_\nabla) \coloneqq a_0 \,\mathrm{Id}_E + a_1 \Theta_\nabla + \cdots + a_m \Theta^m_\nabla \in \mathcal{A}^\bullet_{\mathbb{C}}(X, \mathrm{End}(E)).$$

Prove that $\mathrm{d}\,\mathrm{tr}(f(\Theta_\nabla)) = 0$.

(8) Let ∇ and ∇' be two connections on E. For $t \in [0, 1]$, define $\nabla_t = t\nabla + (1-t)\nabla'$. Prove that ∇_t is a connection and that

$$\frac{\mathrm{d}\Theta_{\nabla_t}}{\mathrm{d}t} = \left[\nabla_t, \frac{\mathrm{d}\nabla_t}{\mathrm{d}t}\right].$$

(9) For every polynomial $f \in \mathbb{C}[t]$, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}\big(f(\Theta_{\nabla_t})\big) = \mathrm{d}\,\mathrm{tr}\left(\frac{\mathrm{d}\nabla_t}{\mathrm{d}t} \wedge f'(\Theta_{\nabla_t})\right).$$

(10) Prove that there exists a differential form $\eta \in \mathcal{A}^\bullet_{\mathbb{C}}(X)$ such that

$$\mathrm{tr}\, f(\Theta_\nabla) - \mathrm{tr}\, f(\Theta_{\nabla'}) = \mathrm{d}\eta.$$

(11) Prove that the Chern classes $c_j(E) \coloneqq [c_j(E, \nabla)] \in H^{2j}_{\mathrm{dR}}(X; \mathbb{C})$ are independent of the choice of ∇.

Exercise 4 (The metric on the unit ball)**.** Let $X = \mathbb{B}^n(0, 1) \subset \mathbb{C}^n$, with $n \geq 1$, be the unit ball, with complex coordinates $z = (z_1, \ldots, z_n)$, where $z_i = x_i + \sqrt{-1}y_i$. Consider the $(1,1)$-form

$$\omega \coloneqq -\frac{\sqrt{-1}}{2}\partial\bar\partial \log(1 - \|z\|^2) = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^n h_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\bar z_j.$$

Let ∇ be the Chern connection with respect to ω on T_X. Let h be the natural Hermitian metric on T_X induced by ω.

(1) Prove that ω is d-closed.

(2) Find a Kähler metric g on X in terms of $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, 1 \leq i \leq n\}$ such that $\omega_g = \omega$.

(3) Assume $n = 1$. Compute the curvature of ∇ in terms of $dz \wedge d\bar z$.

(4) Assume $n = 1$. Compute the curvature forms of the Chern connections on T_X for the metrics associated with the following Kähler forms:

$$\omega_1 \coloneqq \frac{\sqrt{-1}}{2}\partial\bar\partial \log(1 + |z|^2), \qquad \omega_2 \coloneqq \frac{\sqrt{-1}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar z.$$

(5) Let $\langle z, a\rangle = \sum_{i=1}^n z_i \bar a_i$ be the standard Hermitian form on $\mathbb{C}^n$. For each $a \in X$, let P_a be the orthogonal projection onto $\mathbb{C}a$ and let $Q_a = \mathrm{Id} - P_a$. In particular, $P_0 = 0$. Set $s_a = (1 - \|a\|^2)^{1/2}$ and define

$$\Phi_a(z) \coloneqq \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}.$$

Prove that Φ_a is a holomorphic automorphism of X sending 0 to a.

(6) For every $n \geq 1$, every point $x \in X$, and all holomorphic vector fields $s, v \in H^0(X, T_X)$ with $s(x) \neq 0$ and $v(x) \neq 0$, prove that

$$h\big(\Theta_\nabla(v, \bar v)s, s\big)(x) < 0.$$