Exercise sheet 6

Exercise 1. For two integers $n \geq 1$, $d \geq 2$. Let $X_{n,d} \subset \mathbb{P}^{n+1}(k)$ be the n-dimensional Fermat hypersurface of degree $d \geq 2$ defined by the following equation

$$X_0^d + X_1^d + \cdots + X_n^d + X_{n+1}^d = 0,$$

where $[X_0 : X_1 : \cdots : X_n : X_{n+1}]$ is the homogeneous coordinates of $\mathbb{P}^{n+1}(k)$.

(1) Prove that $X_{n,d}$ is nonsingular.

(2) Let $H \subset \mathbb{P}^{n+1}(k)$ be the hyperplane defined by the equation $X_0 = 0$. Prove that $X_{n,d}$ is linearly equivalent to dH as divisors in $\mathbb{P}^{n+1}(k)$.

(3) Let $\iota : X_{n,d} \hookrightarrow \mathbb{P}^{n+1}(k)$ be the natural closed immersion. Show that there exists a natural short exact sequence as following

$$0 \to \mathscr{O}_{\mathbb{P}^{n+1}(k)}(-d) \to \mathscr{O}_{\mathbb{P}^{n+1}(k)} \to \iota_* \mathscr{O}_{X_{n,d}} \to 0.$$

(4) Deduce that $H^q(X_{n,d}, \mathscr{O}_{X_{n,d}}(m)) = 0$ for $1 \leq q \leq n-1$.

(5) Compute $\dim_k H^0(X_{n,d}, \mathscr{O}_{X_{n,d}}(m))$ and $\dim_k H^n(X_{n,d}, \mathscr{O}_{X_{n,d}}(m))$.

(6) Prove that if $X_{1,d}$ is isomorphic to $X_{1,d'}$, then $d = d'$.

Exercise 2 (Invariants of projective varieties)**.** Let X be a nonsingular irreducible projective variety.

(1) (Euler characteristic) Let $\mathscr{F}$ be a coherent sheaf on X. The *Euler characteristic* of $\mathscr{F}$ is defined as

$$\chi(\mathscr{F}) \coloneqq \sum (-1)^i h^i(X, \mathscr{F}).$$

Prove that if $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{Q} \to 0$ is a short exact sequence of coherent sheaves on X, then we have

$$\chi(\mathscr{E}) = \chi(\mathscr{F}) + \chi(\mathscr{Q}).$$

(2) The number $\chi(X, \mathscr{O}_X)$ is called the *Euler characteristic of* X. Determine the Euler characteristic of $\mathbb{P}^n$ and a nonsingular hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d.

(3) (Geometric genus) The *geometric genus* $p_g(X)$ *of* X is defined as $h^0(X, \omega_X)$. Determine the geometric genus of a nonsingular hypersurface in $\mathbb{P}^{n+1}$ of degree d.

(4) (Arithmetic genus) The *arithmetic geneus* $p_a(X)$ *of* X is defined as $(-1)^n(\chi(X, \mathscr{O}_X) - 1)$, where $n = \dim(X)$. Prove that if $\dim(X) = 1$, then we have

$$p_a(X) = h^1(X, \mathscr{O}_X).$$

Derive that we have $p_a(X) = p_g(X)$ if $\dim(X) = 1$.

(5) (Genus-degree formula) Let $C \subset \mathbb{P}^2$ be a nonsingular curve of degree d. Prove that we have

$$p_g(C) = p_a(C) = \frac{(d-1)(d-2)}{2}.$$

In particular, if C and C' are two nonsingular curves in $\mathbb{P}^2$ with degree $d \neq d' \geq 2$, then C is not isomorphic to C'.

(6) Show that the conic C in $\mathbb{P}^2$ defined as $X_0^2 + X_1^2 + X_3^2 = 0$ is isomorphic to $\mathbb{P}^1$.

(7) (Irregularity) The *irregularity* $q(X)$ *of* X is defined as $h^1(X, \mathscr{O}_X)$. Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of dimension $n \geq 2$. Prove that X is regular, i.e., $q(X) = 0$.

Exercise 3 (Blow-up of quadratic cone)**.** Let $X \subset \mathbb{A}^3_k$ be the quadratic cone defined by $X_1X_2 = X_3^2$. Let $\pi : V \to X$ be the blow-up of $\mathbb{A}^3_k$ with centre in the origin, and X' the closure of $\pi^{-1}(X \setminus \{0\})$; i.e. X' is the blow-up of X along 0.

(1) Prove that X' is nonsingular.
(2) Prove that the inverse image of the origin under $X' \to X$ is isomorphic to $\mathbb{P}^1$.

Exercise 4 (Du Val singularity of type D_4)**.** Let $X \subset \mathbb{A}^3_k$ be the affine subvariety defined by the equation $X_1^2 + X_2^3 + X_3^3 = 0$.

(1) Prove that the origin is the only singular point of X.
(2) Determine the blow-up $\sigma : X_1 \to X$ of X along 0.
(3) Prove that there are three singular points of X_1.
(4) Let $\pi : X_2 \to X_1$ be the blow-up of X_1 along the three singular points of X_1. Prove that X_2 is nonsingular.
(5) Draw the dual graph of the composition $f : X_2 \to X_1 \to X$; that is, a graph whose vertexes corresponds to the irreducible components of $f^{-1}(0)$ and two vertexes are joined by a line if they intersects each other.

Exercise 5. Let $\varphi : X \to Y$ be a dominant morphism of projective varieties. Assume that Y is irreducible, and all the fibres $\varphi^{-1}(y)$ for $y \in Y$ are irreducible and of constant dimension n. We want to prove that X is also irreducible.

(1) Prove that φ is surjective and closed.
(2) Prove that $\dim(X) = n + \dim(Y)$.
(3) Let $X = X_1 \cup \cdots \cup X_r$ be the decomposition of X into irreducible components. Prove that there exists a component X_i such that $\varphi(X_i) = Y$.

In what follows we assume that the components X_i such that $\varphi(X_i) = Y$ are those of index $i = 1, \ldots, s$, $1 \leq s \leq r$. We denote the restriction of φ to X_i by φ_i.

(4) Prove there is an $i \leq s$ such that $\dim(X_i) = \dim(X)$. Prove that for such i all the fibres of the maps φ_i are of dimension $\geq n$.
(5) Prove that $X = X_i$ with X_i an irreducible component of X provided in (4).
(6) Deduce that X is irreducible.
(7) Show that the theorem does not hold if the varieties are not assumed to be projective by considering the following example: let X be the union of the origin in $\mathbb{A}^2_k$ and the hyperbole $TW = 1$ and let $\varphi : X \to Y$ be the projection onto the T-axis, where (T, W) are the coordinates of $\mathbb{A}^2_k$.