Exercise sheet 5

Exercise 1 (Stalks of hom-sheaf)**.** Let $X = [0,1]$ with cofinite topology. Let $\underline{\mathbb{Z}}_X$ be the constant sheaf over X with respect to $\mathbb{Z}$.

(1) Let $\mathcal{F}$ be the skycraper sheaf $\mathbb{Z}$ at $0 \in X$. Show that $\mathcal{F}$ admits a natural $\underline{\mathbb{Z}}_X$-module structure, i.e. $\mathcal{F}$ is a sheaf of $\underline{\mathbb{Z}}_X$-modules.

(2) Let $U \subset X$ be a non-empty open subset of X. Show that we have
$$\mathrm{Hom}_{\underline{\mathbb{Z}}_X}(\mathcal{F}|_U, \underline{\mathbb{Z}}_X|_U) = 0.$$

(3) Determine the stalks of $\mathcal{F}$ and $\underline{\mathbb{Z}}_X$ at 0.

(4) Determine the group $\mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{F}_0, \underline{\mathbb{Z}}_{X,0})$. In particular, prove that the natural homomorphism
$$\mathscr{H}\mathrm{om}_{\underline{\mathbb{Z}}_X}(\mathcal{F}, \underline{\mathbb{Z}}_X)_0 \to \mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{F}_0, \underline{\mathbb{Z}}_{X,0})$$
is not surjective.

(5) Let $U = X \setminus \{0\}$ and let $\underline{\mathbb{Z}}_U$ be the constant sheaf over U with respect to $\mathbb{Z}$. Denote by $\mathscr{G}$ the sheaf obtained by extending $\underline{\mathbb{Z}}_U$ by zero (see Exercise 6 in Exercise sheet 1). Prove that $\mathscr{G}$ is a sheaf of $\underline{\mathbb{Z}}_X$-modules.

(6) Determine the stalk $\mathscr{G}_0$ and derive the following
$$\mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathscr{G}_0, \mathscr{G}_0) = 0.$$

(7) Let $V \subset X$ be a non-empty open subset of X. Show that there exists a natural inclusion
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(\mathscr{G}|_V, \mathscr{G}|_V).$$

(8) Show that there exists a natural inclusion
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \hookrightarrow \mathscr{H}\mathrm{om}_{\underline{\mathbb{Z}}_X}(\mathcal{G}, \mathcal{G})_0.$$
In particular, derive that the natural homomorphism
$$\mathscr{H}\mathrm{om}_{\underline{\mathbb{Z}}_X}(\mathcal{G}, \mathcal{G})_0 \to \mathrm{Hom}_{\underline{\mathbb{Z}}_{X,0}}(\mathcal{G}_0, \mathcal{G}_0)$$
is not injective.

Exercise 2 (Ideal sheaves on $\mathbb{P}^n$)**.** Let $X = \mathbb{P}^n$ and $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$ with $\mathfrak{m} = (x_0, \ldots, x_n)$. A homogeneous ideal I in S is called *saturated* if for every element $u \in S$ such that $u\mathfrak{m} \subset I$, we have $u \in I$.

(1) Show that for every homogeneous ideal I in S, there is a unique saturated ideal I^s in S such that $I \subset I^s$ and there is r such that $\mathfrak{m}^r \cdot I^s \subset I$.

(2) Show that if I_1 and I_2 are homogeneous ideals in S, then $\widetilde{I_1} \cong \widehat{I_2}$ if and only if $I_1^s = I_2^s$.

(3) In particular, given any coherent ideal sheaf on $\mathscr{I}$ on $\mathbb{P}^n$, there is a unique saturated ideal I on S such that $\widetilde{I} \cong \mathscr{I}$. Show that this is the unique largest homogeneous ideal I such that $\widetilde{I} \cong \mathscr{I}$, and it is equal to $\Gamma_* \mathscr{I} \subset S$.

(4) Show that if I is a homogeneous radical ideal $\neq \mathfrak{m}$, then I is saturated.

Exercise 3 (Differentials)**.** Let k be an algebraically closed field.

(1) Let $X \coloneqq \{x^2 + y^2 = 1\} \subset k^2$. Assume that k has characteristic $\neq 2$. Prove that dx/y is defines an element in $\Gamma(X, \Omega_X)$ and then deduce that $\Gamma(X, \Omega_X) \cong \mathcal{O}_X(X)[dx/y]$.

(2) Let $X \coloneqq \{x_0^3 + x_1^3 + x_2^3 = 0\} \subset \mathbb{P}^2$ and assume that k has characteristic $\neq 3$. Prove that $\dim \Gamma(X, \Omega_X) = 1$.

(3) Let $\Omega^r_{\mathbb{P}^n} := \wedge^r \Omega_X$. Prove that $\Gamma(\mathbb{P}^n, \Omega^r_{\mathbb{P}^n}) = 0$ for $r > 0$.

Exercise 4 (Divisible groups)**.** An abelian group G is called *divisible* if for any $0 \neq n \in \mathbb{Z}$, the homomorphism $G \to G$ defined by $x \mapsto nx$ is surjective.

(1) Prove that divisible groups are injective in the category of abelian groups.
(2) Prove that $\mathbb{Q}/\mathbb{Z}$ is divisible.
(3) For an abelian group G, we define its dual $\check{G}$ as $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. Prove that there exists a natural injective homomorphism $G \to \check{\check{G}}$.
(4) Let G be a free group. Prove that $\check{\check{G}}$ is injective.
(5) Prove that the category of abelian groups has enough injectives.

Exercise 5 (Cohomologies of cotangent sheaves)**.** In this exercise, we want to compute the cohomologies of the cotangent sheaves of certain nonsingular varieties.

(1) Use the Čech cohomology to show that for a family $\{\mathcal{F}_i\}_{i \in I}$ of quasi-coherent sheaves on an algebraic variety X, we have a natural isomorphism

$$H^q(X, \bigoplus_{i \in I} \mathcal{F}_i) \cong \bigoplus_{i \in I} H^q(X, \mathcal{F}_i).$$

(Hint: use the fact that X is quasi-compact and find a suitable finite affine open covering of X.)

(2) Now we want to compute the cohomologies of $\Omega_{\mathbb{P}^n}$ using the following Euler sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

   (a) Determine the cohomologies of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$ using (1).
   (b) Determine the cohomologies of $\mathcal{O}_{\mathbb{P}^n}$.
   (c) Use the Euler sequence to derive a long exact sequence to determine the cohomologies of $\Omega_{\mathbb{P}^n}$.

(3) Let $i : X \hookrightarrow \mathbb{P}^{n+1}$ be a nonsingular quadric hypersurface. In the following we want to compute the cohomologies of Ω_X. In the proof, you can use the following fact: let $j : Z \hookrightarrow Y$ be a closed variety in an algebraic variety Y and let $\mathcal{F}$ be a coherent sheaf, then we have a natural isomorphism, for each $q \geq 0$,

$$H^q(Z, j^*\mathcal{F}) \cong H^q(Y, \mathcal{F} \otimes j_*\mathcal{O}_Z).$$

   (a) Show that the following sequence of coherent sheaves is exact

$$0 \to \Omega_{\mathbb{P}^n}(-2) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(-2) \to 0.$$

   Then use it to determine the cohomologies of $\Omega_{\mathbb{P}^n}(-2)$.

   (b) Show that the following sequence of coherent sheaves is exact

$$0 \to \Omega_{\mathbb{P}^n} \otimes \mathcal{I}_X \to \Omega_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n} \otimes i_*\mathcal{O}_X \to 0.$$

   Then use it to determine the cohomologies of $i^*\Omega_{\mathbb{P}^n}$.

   (c) Use the following conormal sequence to determine the cohomologies $H^q(X, \Omega_X)$ for $q \leq n - 2$

$$0 \to \mathscr{O}_X(-2) \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0.$$