# Exercise sheet 4

**Exercise 1** ($\mathbb{Q}$-Cartier divisor which is not Cartier). Let $X = \{X_1X_2 = X_3^2\} \subset \mathbb{A}_k^3 \ni (X_1, X_2, X_3)$. Let $D$ be the prime divisor in $X$ defined by the following equation $X_1 = X_3 = 0$.

(1) Find the singular locus of $X$.
(2) Let $U = D(X_2) \subset X$ be the standard open subset of $X$ defined as $X_2 \neq 0$. Find the generators of the ideal of $D \cap D(X_2)$ in $\Gamma(D(X_2), \mathcal{O}_X)$.
(3) Show that the Weil divisor $2D$ is Cartier.
(4) Let $Y \subset \mathbb{A}_k^3 \ni (Y_1, Y_2, Y_3)$ be the variety defined by the following equation $Y_2 = Y_3^2$. Show that $Y$ is nonsingular.
(5) Consider the morphism $\Phi : \mathbb{A}_k^3 \to \mathbb{A}_k^3$ which sends the point $(x_1, x_2, x_3)$ to the point $(x_1, x_1x_2, x_1x_3)$. Prove that the restriction $f := \Phi|_Y$ of $\Phi$ to $Y$ is a birational morphism from $Y$ to $X$.
(6) Let $E = f^{-1}(0)$ be the preimage of the original point in $Y$. Show that $E$ is a prime divisor in $Y$ and find the generators of the ideal of $E$ in $\Gamma(Y, \mathcal{O}_Y)$.
(7) Show that $f|_{Y\setminus E} : Y \setminus E \to X \setminus D$ is an isomorphism.
(8) Calculate the pull-back $f^*(2D)$ and prove that $D$ is not Cartier.
(9) Let $D' \subset X$ be the prime divisor in $X$ defined by the equation $X_2 = X_3 = 0$. Show that $2D'$ is Cartier.
(10) Calculate the pull-back $f^*(2D')$ and then prove that $D'$ is not Cartier.
(11) Prove that $D + D'$ is Cartier.

**Exercise 2** (Class group of affine varieties). Let $X = \{X_1X_2 = X_3^2\} \subset \mathbb{A}_k^3$ be the variety as in Exercise 1 and we follow the same notations as in Exercise 1.

(1) Show that we have $\Gamma(U, \mathcal{O}_X) = k[X_1, X_1^{-1}, X_3]$, where $U = X \setminus D$.
(2) Show that $U$ is a nonsingular irreducible affine variety and $\Gamma(U, \mathcal{O}_U)$ is a UFD.
(3) Show that $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$ and $D$ is a generator for it.
(4) Prove that $X$ is $\mathbb{Q}$-factorial.

**Exercise 3** (Blowing-up points of affine spaces). For $n \geq 1$, let $X \subset \mathbb{A}_k^{n+1} \times \mathbb{P}^n(k)$ be the closed subvariety defined by the following equation

$$X_iY_j = X_jY_i, \quad 0 \leq i, j \leq n.$$

where $(X_0, \ldots, X_n)$ is the coordinate of $\mathbb{A}_k^{n+1}$ and $[Y_0 : \cdots : Y_n]$ is the homogeneous coordinate of $\mathbb{P}^n(k)$. Denote by $\pi : X \to \mathbb{A}_k^{n+1}$ the first projection. Let $0 \in \mathbb{A}_k^{n+1}$ be the original point and denote by $E$ the preimage $\pi^{-1}(0)$.

(1) Prove that $X$ is irreducible and nonsingular.
(2) Prove that $E \cong \mathbb{P}^n(k)$.
(3) Prove that $\pi|_{X\setminus E} : X \setminus E \to \mathbb{A}_k^{n+1} \setminus \{0\}$ is an isomorphism.
(4) For any integer $a \in \mathbb{Z}$, prove that the divisor $aE$ is principal if and only if $a = 0$. (Hint: otherwise, show that $aE$ is given by $\mathrm{div}(f \circ \pi)$, where $f$ is a regular function on $\mathbb{A}_k^{n+1}$ nonwhere vanishing).
(5) Determine the class group $\mathrm{Cl}(X)$.

The morphism $\pi : X \to \mathbb{A}_k^n$ is called the *blowing-up of* $\mathbb{A}_k^n$ *along* 0.

**Exercise 4** (Automorphism group of projective spaces)**.** Recall that $\mathrm{GL}(n+1,k)$ is the group of all invertible $(n+1)\times(n+1)$-matrices over $k$ with the operation of matrices multiplication.

(1) Show that for every element $A \in \mathrm{GL}(n+1,k)$ induces a natural isomorphism of $\mathbb{P}^n(k)$.
(2) Show that the kernel of the action of $\mathrm{GL}(n+1,k)$ on $\mathbb{P}^n(k)$ is the subgroup $\{cI \mid c \in k^*\}$ of central scalar matrices. Here we recall the kernel is defined to be the subset of $\mathrm{GL}(n+1,k)$ consisting of elements which act on $\mathbb{P}^n(k)$ as identity. Denote by $\mathrm{PGL}(n+1,k)$ the quotient group. We call $\mathrm{PGL}(n+1,k)$ the *projective general linear group.*
(3) Let $g : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ be an automorphism of $\mathbb{P}^n(k)$. Show that $g$ induces a linear isomorphism of the $k$-vector space $\Gamma(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)}(1))$.
(4) An automorphism of $\mathbb{P}^n(k)$ is called a *projective transformation* if it is given by an element in $\mathrm{PGL}(n+1,k)$. Show that every automorphism of $\mathbb{P}^n(k)$ is a projective transformation.

**Exercise 5.** Let $\varphi\colon \mathbb{P}^n \to \mathbb{P}^N$ be the $d$-th Veronese embedding, given by $[x_0 : \cdots : x_n] \mapsto [x^\alpha]_{\alpha\in A}$, where

$$A = \left\{(\alpha_0,\dots,\alpha_n) \in \mathbb{Z}^{n+1}_{\geq 0} \mid \sum \alpha_i = d\right\}, \quad x^\alpha = x_0^{\alpha_0}\dots x_n^{\alpha_n}.$$

(1) Prove that $\varphi$ is injective.
(2) Consider the ring homomorphism

$$\theta\colon k[(Y_\alpha)]_{\alpha\in A} \longrightarrow k[X_0,\dots,X_n]$$

defined by $\theta(Y_\alpha) = X^\alpha$. Set $I = \ker(\Theta)$ and consider $V = V_p(I)$ (the Veronese variety).

Prove that $I$ is a homogeneous ideal and $\varphi(\mathbb{P}^n) \subset V$.
(3) Consider $\alpha,\beta,\gamma,\delta \in A$. Prove that if $\alpha+\beta=\gamma+\delta$, then $Y_\alpha Y_\beta - Y_\gamma Y_\delta \in I$.
(4) Prove that the open subset $D^+(Y_{(i)})$ cover $V$, where $(i) = (\alpha_j)$ with $\alpha_j = \delta_{ij}d$.
(5) Define $\psi\colon D^+(Y_{(i)}) \cap V \to D^+(X_i)$ by the formula

$$\psi((y_\alpha)) := (y_{(i,0)}, y_{(i,1)}, \dots, y_{(i)}, \dots, y_{(i,n)}),$$

where $(i,j) = (\alpha_k)$ with $\alpha_k = 0$ if $k \neq i,j$, $\alpha_i = d-1$ and $\alpha_j = 1$.

Prove that $\varphi$ and $\psi$ are mutually inverse morphisms on the open sets in question.
(6) Prove that $\varphi$ gives an isomorphism from $\mathbb{P}^n$ to the Veronese variety $V$.

**Exercise 6.** Let $\varphi\colon X \to Y$ be a morphism of varieties. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a $\mathscr{O}_X$-module (resp. a $\mathscr{O}_Y$-module).

(1) Prove that there are natural morphisms $\varphi^*\varphi_*\mathscr{F} \to \mathscr{F}$ and $\mathscr{G} \to \varphi_*\varphi^*\mathscr{G}$. Deduce the following so called adjunction formula:

$$\mathrm{Hom}_{\mathscr{O}_X}(\varphi^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, \varphi_*\mathscr{F}).$$

(2) Assume that $\varphi$ is a locally closed immersion.
 (a) Prove that $\varphi^*\varphi_*\mathscr{F}$.
 (b) Let $\mathscr{F}'$ be another $\mathscr{O}_X$-module. Prove that

$$\mathrm{Hom}_{\mathscr{O}_Y}(\varphi_*\mathscr{F}, \varphi_*\mathscr{F}') \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}').$$

 (c) Prove that $\varphi_*$ and $\varphi_*$ provide an equivalence of categories between $\mathscr{O}_X$-modules and $\mathscr{O}_Y$-modules of the form $\varphi_*\mathscr{F}$. This allows us to identify $\mathscr{F}$ to $\varphi_*\mathscr{F}$.