Exercise sheet 3

Throughout we denote by k an algebraically closed field.

Exercise 1 (Properties of projective varieties). (1) Prove that $\mathbb{P}^n$ is irreducible.

(2) Prove that a graded ring R is an integral domain if and only if for all homogeneous elements $f, g \in R$ with $fg = 0$ we have $f = 0$ or $g = 0$.
(3) Show that a projective variety X is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.
(4) Show by example that the homogeneous coordinate ring of a projective variety is not invariant under isomorphisms, i.e. that there are isomorphic projective varieties X, Y such that the rings $S(X)$ and $S(Y)$ are not isomorphic.
(5) Let $X \subset \mathbb{P}^n$ be a projective variety and let $S(X)$ be its homogeneous coordinate ring. For any non-zero homogeneous element $f \in S(X)$, prove that there exists a canonical isomorphism
$$\mathscr{O}_X(D^+(f)) \cong S(X)_{(f)}.$$

Exercise 2 (Singular points of projective hypersurfaces). Let k be an algebraically closed field of characteristic zero. Let $\mathbb{P}^n$ be the n-dimensional projective space over k. Recall that a *hypersurface* of $\mathbb{P}^n$ is a projective subvariety of $\mathbb{P}^n$ defined by a non-zero homogeneous polynomial. Moreover, given a hypersurface X in $\mathbb{P}^n$, then there exists a unique reduced polynomial F such that the homogenous ideal of X is generated by F. Then X is said to be defined by the polynomial F.

(1) Prove that the singular points of a hypersurface $X \subset \mathbb{P}^n$, which is defined by a homogeneous polynomial $F(x_0, \cdots, x_n) = 0$, are determined by the system of equations
$$F(x_0, \cdots, x_n) = 0 \quad \text{and} \quad \frac{\partial F}{\partial X_i}(x_0, \cdots, x_n) = 0 \text{ for } i = 0, \cdots, n.$$
(2) Prove that we have the following equality, which is known as *Euler's Theorem.*
$$\deg(F) \cdot F = \sum_{i=0}^{n} X_i \frac{\partial F}{\partial X_i}.$$
(3) Determine the singular points of the Steiner surface in $\mathbb{P}^3$:
$$x_1^2 x_2^2 + x_2^2 x_0^2 + x_0^2 x_1^2 - x_0 x_1 x_2 x_3 = 0.$$
(4) Prove that if a hypersurface $X \subset \mathbb{P}^n$ contains a linear subspace L of dimension $r \geq n/2$, then X is singular. (Hint: choose the coordinate system so that L is given by $x_{r+1} = \cdots = x_n = 0$, write out the equation of X and look for singular points contained in L.)
(5) Let $p \in \mathbb{P}^n$ be a point and let $L_1, \ldots, L_n$ be n linear forms in $k[x_0, \ldots, x_n]$ such that $V(L_1, \ldots, L_n) = \{p\}$. Let $\pi_p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be the rational map defined as following:
$$[x_0 : \cdots : x_n] \mapsto [L_1(x_0, \ldots, x_n) : \cdots : L_n(x_0, \ldots, x_n)].$$
Show that π_p is a well-defined morphism over $\mathbb{P}^n \setminus \{p\}$.
(6) Let $p \in \mathbb{P}^n$ be a point. A *cone over* p is the closure of the preimage $\pi_p^{-1}(Y)$ for a projective subvariety $Y \subset \mathbb{P}^{n-1}$. Prove that a hypersurface of degree two with a singular point is a cone. Here the degree of hypersurface is defined as the degree of the defining polynomial. (Hint: consider the projection from a singular point).

(7) Let X be an irreducible hypersurface of degree 3. Assume that the singular locus of X contains two distinct points p and q. Prove that the line joining p and q is contained in X. Here a line means a projective subspace of dimension one in $\mathbb{P}^n$.

Exercise 3 (Projective tangent spaces)**.** Let $X \subset \mathbb{P}^n$ be an irreducible projective variety and let $p \in X$ be a point. Show that the following definitions of the "**projective tangent space**" of X at p are equivalent:

(1) The closure in $\mathbb{P}^n$ of the tangent space to the affine variety $X \cap U_i$ at p, where U_i is any standard affine chart containing p.

(2) The projective linear subspace corresponding to the subspace of k^{n+1}, which is the kernel of the $r \times (n+1)$ scalar matrix

$$J = \left(\frac{\partial F_i}{\partial X_j}(x_0, \ldots, x_n) \right),$$

where $\{F_1, \ldots, F_r\}$ is a family of homogeneous generators of the homogeneous ideal $V(X)$ and $(x_0, \cdots, x_n) \in k^{n+1}$ is an arbitrary point representing p.

(3) The projective linear subspace corresponding to the linear subspace $T_{\widetilde{p}}\widetilde{X}$ of k^{n+1}, where $\widetilde{X} \subset k^{n+1}$ is the affine cone of X and $\widetilde{p} \in \widetilde{X}$ is any point representing p.

Exercise 4 (Closed points of schemes)**.** (1) Let A be the coordinate ring of an affine variety over an algebraically closed field. Prove that the subset of closed points in $\operatorname{Spec}(A)$ is dense in $\operatorname{Spec}(A)$.

(2) Give an example to show that this is no longer true for general schemes.

Exercise 5 (Nilpotent elements and tangent spaces)**.** (1) Prove that a scheme X is reduced if and only if there is an open cover of X by affine schemes $U_i = \operatorname{Spec}(R_i)$ such that every ring R_i has no nilpotent elements, and if and only if $\mathscr{O}_{X,x}$ has no nilpotent elements for any point $x \in X$.

(2) For $n \in \mathbb{Z}_{>0}$, an *n-fold point* or *fat point* over k is a scheme over k of the form $\operatorname{Spec}(R)$ that contains only one point, and such that R is a k-algebra of vector space dimension n over k.

(a) Show that every double point over k is isomorphic to $k[x]/\langle x^2 \rangle$.

(b) Find two non-isomorphic triple points over k and describe them geometrically?

(3) Let x be a closed point on a variety X over k, and denote by $D := \operatorname{Spec}(k[x]/\langle x^2 \rangle)$ the double point. The *Zariski tangent space* $T_{X,x}$ of X at x is defined as $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$. Show that $T_{X,x}$ can be canonically identified with the set of morphisms $D \to X$ that map the unique point of D to x.

Exercise 6. Let X be a variety and let $\{U_i\}_{i \in I}$ be an open covering of X. Set $U_{ij} = U_i \cap U_j$.

(1) Let E and F be vector bundles over X with transition functions $g_{ij} : U_{ij} \to \mathrm{GL}_r(k)$ and $h_{ij} : U_{ij} \to \mathrm{GL}_s(k)$. Write down the transition functions of the following vector bundles in term of g_{ij} and h_{ij}:

$$E \otimes F, \quad \operatorname{Hom}(E, F), \quad E^*, \quad \wedge^k E, \quad \det(E), \quad , S^k E.$$

(2) Let $X = \mathbb{P}^n$ and let $U_i = D^+(x_i)$ be the standard open subsets. Write down the transition functions for T_X and $\mathscr{O}_{\mathbb{P}^n}(m)$, $m \in \mathbb{Z}$. Deduce that $K_{\mathbb{P}^n}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^n}(-n-1)$.

(3) Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles over X. Prove that there exists a canonical isomorphism

$$\det(E) \cong \det(E') \otimes \det(E'').$$