Exercise sheet 2

Exercise 1 (Maximal spectrum and Zariski topology)**.** Let A be a commutative ring with identity element 1. Let $X = \mathrm{MaxSpec}(A)$ be the set of all maximal ideals in A. For an ideal $\mathfrak{a}$ of A, we define $V(\mathfrak{a})$ to be the subset of X consisting of all maximal ideals of A containing $\mathfrak{a}$.

(1) Show that $V(0) = X$ and $V(1) = \emptyset$.
(2) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, show that

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b}).$$

(3) If $\{\mathfrak{a_i}_{i\in I}\}$ is a family of ideals in A, show that

$$V\left(\sum_{i\in I} \mathfrak{a}_i\right) = \bigcap_{i\in I} V(\mathfrak{a}_i).$$

These properties show that all $V(\mathfrak{a})$ satisfy the axioms for closed subsets in a topological space. We call it the *Zariski topology* on the *maximal spectrum* $X = \mathrm{MaxSpec}(A)$ of A.

Let k be an algebraically closed field and let $S \subset \mathbb{A}^n_k$ be an affine algebraic set. Let $I(S)$ be the ideal of S and denote by A the coordinate ring $k[x_1, \dots, x_n]/I(S)$.

(1) (Weak Nullstellensatz) Let $x \in S$ be a point and let $\mathfrak{m}_x \subset A$ be the ideal consisting of elements which vanish on x. Show that $\mathfrak{m}_x$ is a maximal ideal of A and the map $\Phi : S \to X = \mathrm{MaxSpec}(A)$ by sending x to $\mathfrak{m}_x$ is a bijection.
(2) Show that Φ is a homeomorphism with respect to the Zariski topologies on S and X.
(3) Give an example to show that Φ is not surjective if k is not algebraically closed, e.g. $k = \mathbb{R}$.

As a consequence, all the datum (i.e. points, topology, structure sheaf) of an affine algebraic variety X defined over *an algebraically closed field* k is determined by $A = \mathscr{O}_X(X)$, the ring of regular functions on X

Exercise 2. A ring A is said to be *connected* if every idempotent in A is trivial (i.e. if every element e in A such that $e^2 = e$ is equal to 0 or 1).

(1) Prove that every integral domain is connected.
(2) If A is the direct product of two non-trivial rings, prove that A is not connected.
(3) Conversely, if A possesses a non-trivial idempotent e, prove that $A \cong A/(e) \times A/(1-e)$.
(4) Let V be an affine algebraic variety over an algebraically closed field k. Prove that V is connected (in the Zariski topology) if and only if $\mathscr{O}_V(V)$ is connected. (Hint: If V has two connected components, start by finding a function which is 0 on one and 1 on the other.) Is this still the case if k is not algebraically closed?

Exercise 3 (Noether's normalization theorem)**.** The aim of this exercise is to prove the following result:

Theorem 0.1. *Let k be a field (maybe not algebraically closed) and let A be a k-algebra of finite type which is an integral domain. Set $K = \mathrm{Frak}(A)$ and n the transcendental degree of K over k. Then there exist elements $x_1, \dots, x_n \in A$ algebraically independent over k such that A is integral over $k[x_1, \dots, x_n]$.*

(1) Write A as a quotient $lk[Y_1, \ldots, Y_m]/I$. Prove that $m \geq n$ and prove the theorem when $m = n$.
(2) Assume $m > n$. Let $y_1, \ldots, y_m$ be the images of variables Y_i in A. Prove that they satisfy an algebraic equation $F(y_1, \ldots, y_m) = 0$, where F is a non-zero polynomial with coefficients in k.
(3) Choose positive integers $r_2, \ldots, r_m$, and set

$$z_2 = y_2 - y_1^{r_2}, \ldots, z_m = y_m - y_1^{r_m}.$$

Prove that $y_1, z_2, \ldots, z_m$ also satisfy a non-trivial algebraic equation with coefficients in k. (Hint: Prove that, for large enough r_i with large enough growth (i.e. $0 \ll r_2 \ll \cdots \ll r_m$)), y_1 is integral over the subring of A generated by the elements z_i.
(4) Complete the proof the theorem by induction on m.

Exercise 4 (Basic properties of separatedness)**.** Let Y be a separated prevariety.
(1) Any subvarieties of Y are separated.
(2) The intersection of two open affine subvarieties of Y is again an open affine subvariety.
(3) Let $f : X \to Y$ be a morphism from a prevariety X. Prove that the graph of f, $G(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$.

Exercise 5 (Basic properties of completeness)**.** Let X be a complete algebraic variety.
(1) Let $f : X \to Y$ be a morphism to an algebraic variety. Then $f(X)$ is complete.
(2) If Y is a complete algebraic variety. Then $X \times Y$ is again complete.
(3) If $Y \subsetneq X$ is a closed subvariety, then Y is complete.
(4) If X is affine, then $\dim(X) = 0$. In particular, the affine space $\mathbb{A}_k^n$ is not complete if $n \geq 1$.

Exercise 6. Find the singular points of the following varieties and say whether or not they are irreducible.
(1) In $\mathbb{P}^2$:

$$V(XY^4 + YT^4 + XT^4), \quad V(X^2Y^3 + X^2T^3 + Y^2T^3)$$
$$V(X^n + Y^n + T^n),\ n > 0,$$
$$V((X^2 - YT)^2 + Y^3(Y - T)), \quad V(2X^4 + Y^4 - TY(3X^2 + 2Y^2) + Y^2T^2),$$
$$V(Y^2T^2 + T^2X^2 + X^2Y^2 - 2XYT(X + Y + T)).$$

(2) In $\mathbb{P}^3$:

$$V(XY^2 - Z^2T), \quad V(XYT + X^3 + Y^3),$$
$$V(XY^2 - Z^2T), \quad V(XYZ + XYT + XZT + YZT),$$
$$V(XT - YZ, YT^2 - Z^3, ZX^2 - Y^3).$$

Exercise 7 (Dual curves)**.** Let $F(X_0, X_1, X_2)$ be the equation of an irreducible curve $X \subset \mathbb{P}^2$. Consider the rational map $\varphi\colon X \dashrightarrow \mathbb{P}^2$ given by the formulas:

$$u_i = \frac{\partial F}{\partial X_i}(x_0, x_1, x_2), \quad i = 0, 1, 2.$$

(1) Prove that $\varphi(X)$ is a point if and only if X is a line.
(2) Prove that if X is a conic, then so is $\varphi(X)$.
(3) Find the dual curve of $X_0^3 + X_1^3 + X_2^3 = 0$.

If X is not a line, the image $\varphi(X)$ is called the *dual curve* of X.