Exercise sheet 1

Exercise 1 (stalks of presheaves)**.** Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X and let $x \in X$ be a point.

(1) Show that there exists a natural abelian group struction on $\mathscr{F}_x$ such that for any open neighbourhood U of x, the restriction map $\varphi_{U,x} : \mathscr{F}(U) \to \mathcal{F}_x$ is a homomorphism of abelian groups.
(2) Prove that $\mathscr{F}_x \cong \mathscr{F}_x^+$.

Exercise 2 (morphisms of sheaves)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves of abelian groups on a topological space X and let $\varphi : \mathscr{G} \to \mathscr{F}$ be a morphism between presheaves.

(1) Show that the germ $\varphi_x : \mathscr{G}_x \to \mathscr{F}_x$ is a well-defined homomorphism of abelian groups.
(2) Assume furthermore that both $\mathscr{F}$ and $\mathscr{G}$ are sheaves. Prove that the kernel $\ker(\varphi)$ is again a sheaf.

Exercise 3 (image of sheaves)**.** In this exercise, we construct an example $\varphi : \mathscr{G} \to \mathscr{F}$ of morphisms of sheaves such that $\mathrm{preim}(\varphi)$ is not sheaf.

(1) Show that $\mathrm{preim}(\varphi)$ is a presheaf.
(2) Let $X = \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$. For any open subset $U \subset X$, we define

$$\mathscr{G}(U) = \{s : U \to \mathbb{C} \mid s \text{ is continuous}\},$$
$$\mathscr{F}(U) = \{s : U \to \mathbb{C}^* \mid s \text{ is continuous}\}.$$

Prove the $\mathscr{G}$ and $\mathscr{F}$ are sheaves on X. (Hint: the group structions on $\mathscr{F}(U)$ and $\mathscr{G}(U)$ are different!)
(3) For any open subset $U \subset X$, we define a map $\varphi_U : \mathscr{G}(U) \to \mathscr{F}(U)$ as following:

$$\varphi_U(s) \coloneqq \exp(s), \forall s \in \Gamma(U, \mathscr{G}).$$

Show that $\varphi : \mathscr{G} \to \mathscr{F}$ is a morphism of sheaves.
(4) Let $x \in X$ be a point and let $t_x \in \mathscr{F}_x$ be a stalk. Show that there exists an element $s_x \in \mathscr{G}_x$ such that $\varphi_x(s_x) = t_x$. In particular, the morphism φ is surjective. (Hint: one may choose a small open neighbourhood U of x such that $t(U)$ is contained in a single-valued domain of $\mathrm{In}(z)$.)
(5) Show that the map $\mathscr{G}(X) \to \mathscr{F}(X)$ is not surjective.

This example shows why $\mathrm{preim}(\varphi) \neq \mathrm{im}(\varphi)$ in general: given a section $t \in \mathscr{F}(U)$, it may happen that locally at the neighbourhood of every point $x \in U$, we can find a preimage of t in $\mathscr{G}$. However, these local sections of $\mathscr{G}$ can NOT be glued to be a section of $\mathscr{G}(U)$.

Exercise 4 (Skyscraper sheaf)**.** Let X be a topological space and let A be an abelian group. Fix a point $x \in X$. For an open subset $U \subseteq X$, define

$$\mathscr{F}(U) = \begin{cases} A, & \text{if } x \in U \\ 0, & \text{if } x \notin U. \end{cases}$$

(1) Prove that $\mathscr{F}$ is a sheaf.
(2) Find the stalks of $\mathscr{F}$.

Exercise 5 (Supports of sections). Let $\mathscr{F}$ be a sheaf on a topological space X. Let U be an open subset of X and let $s \in \mathscr{F}(U)$ be a section of $\mathscr{F}$ over U. The *support* of s is defined as

$$\mathrm{Supp}(s) \coloneqq \{x \in U \mid s_x \neq 0\}.$$

(1) Prove that $\mathrm{Supp}(s)$ is a closed subset of U.
(2) For $s, t \in \mathscr{F}(U)$, we have $\mathrm{Supp}(s+t) \subset \mathrm{Supp}(s) \cup \mathrm{Supp}(t)$;
(3) If $V \subseteq U$ is an open subset, then $\mathrm{Supp}(s|_V) = \mathrm{Supp}(s) \cap V$.
(4) For a morphism $\varphi : \mathscr{F} \to \mathscr{G}$, we have $\mathrm{Supp}(\varphi_U(s)) \subset \mathrm{Supp}(s)$.

Exercise 6 (extending a sheaf by zero). Let U be an open subset of a topological space X, and let $\mathscr{F}$ be a sheaf on U. Denote by $j : U \to X$ the inclusion. We define $j_!\mathscr{F}$ to be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subset U$, $V \mapsto 0$ otherwise. We call $j_!\mathscr{F}$ the sheaf obtained by *extending $\mathscr{F}$ by zero* outside U.

(1) Find the stalks of $j_!\mathscr{F}$.
(2) Compare $j_!\mathscr{F}$ with the direct image $j_*\mathscr{F}$.
(3) Similar to the support of a section defined in Exercise 5, we can define the support of a sheaf $\mathscr{G}$ on a topological space X as

$$\mathrm{Supp}(\mathscr{G}) \coloneqq \{x \in X \mid \mathscr{G}_x \neq 0\}.$$

Give an example to show that the support of a sheaf may be NOT a closed subset of X.

Exercise 7 (Base change). Let $f\colon X \to Y$ be a continuous map between topological spaces, $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups on Y and X, respectively.

(1) Prove that there exists a canonical isomorphism of abelian groups

$$\mathrm{Hom}(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F}, f_*\mathscr{G}).$$

(2) Give canonical morphisms $\mathscr{F} \to f_*f^{-1}\mathscr{F}$ and $f^{-1}f_*\mathscr{G} \to \mathscr{G}$.
(3) Assume that X is a closed subset of Y and f is the canonical inclusion. Prove that the morphism $f^{-1}f_*\mathscr{G} \to \mathscr{G}$ is an isomorphism and give an example to show that the morphism $\mathscr{F} \to f_*f^{-1}\mathscr{F}$ is not an isomorphism in general.