Exercise sheet 3

Exercise 1 (Properties of Hodge $*$-operator)**.** Let (V, g) be a $\mathbb{R}$-vector space of dimension k equipped with an inner product g. Let $e_1, \ldots, e_k$ be an orthonormal basis of V.

(1) Prove that the vectors $e_{i_1} \wedge \cdots \wedge e_{i_p}$ ($i_1 < \cdots < i_p$) form an orthonormal basis of $\wedge^p V$.
(2) Prove that the Hodge $*$-operator is self-ajoint up to sign: for any $v \in \wedge^p V$ and $v' \in \wedge^{k-p} V$, we have $g(v, *v') = (-1)^{p(k-p)} g(*v, v')$.
(3) Prove that $*^2(v) = (-1)^{p(k-p)} v$ for any $v \in \wedge^p V$.
(4) Prove that $*$ is an isometry, i.e. $g(v, v') = g(*v, *v')$ for any $v, v' \in \wedge^p V$.
(5) For any $1 \leq i_1 < \cdots < i_p \leq k$ and $1 \leq m \leq n$, prove that we have

$$*(e_{i_m} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_p})) = (-1)^{k(p-1)} e_m^* \lrcorner e_{i_1} \wedge \ldots e_{i_p},$$

where e_m^*s are the dual basis of V^* and we define

$$e_m^* \lrcorner e_{i_1} \wedge \ldots e_{i_p} := \begin{cases} 0 & \text{if} \quad m \notin \{i_1, \ldots, i_p\}; \\ (-1)^{r-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_p} & \text{if} \quad m = i_r. \end{cases}$$

Exercise 2 (Standard Kähler metric)**.** Let $X = \mathbb{C}^n$ be the n-dimensional complex affine space with holomorphic coordinates $(z_1, \ldots, z_n)$. Write $z_i = x_i + \sqrt{-1} y_i$ for any $1 \leq i \leq n$. Then $(x_1, y_1, \ldots, x_n, y_n)$ is the real coordinates of $\mathbb{C}^n = \mathbb{R}^{2n}$. Let $\omega := \sqrt{-1} \sum dz_i \wedge d\bar{z}_i$.

(1) Prove that ω is the fundamental form of an inner product g on $\mathbb{R}^{2n}$ which is compatible with the natural complex structure and write down g.
(2) Find an orthonormal basis of $(T_X^{1,0})^* == \Omega_X^{1,0}$ and $(T_X^{0,1})^* = \Omega_X^{0,1}$ with respect to $\widetilde{h}$, where $\widetilde{h}$ is the Hermitian metric induced by g, i.e. $\widetilde{h}(w, w') := g_{\mathbb{C}}(w, \bar{w'})$.
(3) Deduce an orthonormal basis of $\Omega_X^{p,q}$.
(4) Let $I, J \subset \{1, \ldots, n\}$ and $1 \leq k \leq n$. Prove

$$*(dz_k \wedge *(dz_I \wedge d\bar{z}_J)) = \frac{\partial}{\partial \bar{z}_k} \lrcorner dz_I \wedge d\bar{z}_J \quad \text{and} \quad *(d\bar{z}_k * (dz_I \wedge d\bar{z}_J)) = \frac{\partial}{\partial z_k} \lrcorner dz_I \wedge dz_J.$$

Exercise 3 (Kähler forms and Hopf manifolds)**.** Let (X, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$.

(1) For any $1 \leq k \leq n$, prove that $\omega^k \in \mathcal{A}_{\mathbb{R}}^{2k}(X)$ is d-closed.
(2) Prove that $*\omega$ is closed and deduce that ω is harmonic.
(3) Prove that $[\omega^k] \in H^{2k}(X, \mathbb{R})$ is non-zero.
(4) Deduce $h^{k,k} := \dim_{\mathbb{C}} H^{k,k}(X) > 0$.
(5) Let λ be a complex number such that $0 < |\lambda| < 1$. Defined a group action

$$\mathbb{Z} \times (\mathbb{C}^n - \{0\}) \longrightarrow \mathbb{C}^n - \{0\}, (m, z) \longmapsto \lambda^m z.$$

Denote by H the quotient $(\mathbb{C}^n - \{0\})/\mathbb{Z}$. Show that H admits the structure of a complex manifold and is homeomorphic to $S^{2n-1} \times S^1$. (**Hint**: Note that $\mathbb{C}^n - \{0\}$ is diffeomorphic to $S^{2n-1} \times \mathbb{R}^+$.) This complex manifold H is called the n-dimensional *Hopf manifold*.
(6) Prove that the Hopf manifolds are not Kähler if $n \geq 2$.

Exercise 4 (Complex tori)**.** Let $\Gamma \subset \mathbb{C}^n$ be a free abelian discrete subgroup of order $2n$; in other words, Γ is freely generated by an $\mathbb{R}$-basis of $\mathbb{C}^n$ such that $\Gamma \cong \mathbb{Z}^n$. Denote by X the quotient space $\mathbb{C}^n/\Gamma$.

(1) Prove that X admits the structure of a complex manifold, called the *complex torus*.
(2) Let g be the standard Kähler metric on $\mathbb{C}^n$. Show that g descends to a Hermitian metric $\widetilde{g}$ on X.
(3) Show that $\widetilde{g}$ is Kähler.
(4) For $n = 1$, prove that X is isomorphic to a Hopf manifold.

Exercise 5 (Hodge numbers of $\mathbb{P}^n$)**.** We compute the Hodge numbers $h^{p,q}(\mathbb{P}^n)$ of the n-dimensional complex projective spaces. Denote by $[z_0 : \cdots : z_n]$ the homogeneous coordinates of $\mathbb{P}^n$.

(1) Prove that $\mathbb{P}^1$ is homeomorphic to S^2 and deduce the cohomologies of $\mathbb{P}^1$.
(2) Let U be the open subset of $\mathbb{P}^n$ defined by $z_0 \neq 0$. Prove that U is isomorphic to $\mathbb{C}^n$.
(3) Let V be the open subset $\mathbb{P}^n - \{[1 : 0 : \cdots : 0]\}$. Prove that $V \cap U$ is isomorphic to $\mathbb{C}^n - \{0\}$ and is homeomorphic to $\mathbb{R}^{2n} - \{0\}$.
(4) Deduce the cohomologies of $V \cap U$.
(5) Define a map $G\colon V \times [0, 1] \to V$ as
$$[z_0 : z_1 : \cdots : z_n] \longmapsto [tz_0 : z_1 : \cdots : z_n]$$
Show that the map is a well-defined continuous map.
(6) Denote by $Z \subset V$ the closed subset defined by $z_0 = 0$. Prove that Z is isomorphic to $\mathbb{P}^{n-1}$.
(7) Prove that the map G is a homotopy of the retraction of V to Z as follows:
$$V \longrightarrow Z, \quad [z_0 : z_1 : \cdots : z_n] \longmapsto [z_1 : \cdots : z_n].$$
(8) Deduce isomorphisms $H^k(V) \cong H^k(Z)$ for any k.
(9) Recall the Mayer–Vietoris sequence in algebraic topology for singular cohomologies.
(10) Prove that $b_k(\mathbb{P}^n) = 1$ if k is even and $b_k(\mathbb{P}^n) = 0$ if k is odd.
(11) Apply the Hodge decomposition to show $h^{p,q}(\mathbb{P}^n) = \delta_{pq}$ for any $1 \leq p, q \leq n$.

Exercise 6 (Holomorphic forms)**.** Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Denote by Ω_X^p the holomophic vector bundle with underlying complex vector bundle $\Omega_X^{p,0}$. If $n = 2$, then we call X a *compact complex surface.*

(1) Let α be a holomorphic p-forms on X, i.e. $\alpha \in H^0(X, \Omega_X^p)$ is a holomorphic section of Ω_X^p. Prove that α is $\bar{\partial}$-harmonic.
(2) If X is Kähler, then any holomorphic p-form α on X is d-closed.
(3) Let α be a holomorphic 1-form on a compact complex surface X. Prove that
$$\int_X d\alpha \wedge d\bar{\alpha} = 0.$$
(4) Prove that any holomorphic 1-form on a compact complex surface is d-closed.
(5) Prove that any holomorphic 2-form on a compact complex surface is d-closed.
(6) Give an example of holomorphic forms on $\mathbb{C}^n$, $n \geq 2$, which is not d-closed.

Exercise 7 (Properties of Lefschetz operators)**.** Let V be a real $2n$-dimensional vector space equipped with a complex structure J. Let g be an inner product on V, which is compatible with J.

(1) Prove that $[L^i, \Lambda](\alpha) = i(k - n + i - 1)L^{i-1}(\alpha)$ for any $\alpha \in \wedge^k V_{\mathbb{C}}^*$. (**Hint**: Use the relation $[L, \Lambda] = H$ and then argument by induction on i.)
(2) Prove that $L^k\colon \wedge^{n-k} V_{\mathbb{C}}^* \to \wedge^{n+k} V_{\mathbb{C}}^*$ is an isomorphism for $k \leq n$.
(3) Deduce that $L\colon \wedge^k V_{\mathbb{C}}^* \to \wedge^{k+2} V_{\mathbb{C}}^*$ is injective for $k \leq n$ and is surjective for $k \geq n$.
(4) State the similar results for Λ and then prove them.
(5) Prove that $P^{n-k} = \ker(L^{k+1})$ for $k \leq n$.