University of Chinese Academy of Sciences
Pure Mathematics

Fall 2024
Topics in Complex Geometry II

Exercise sheet 2

Exercise 1 (Exact sequence of sheaves)**.** Let $\mathscr{F}$ and $\mathscr{E}$ be a morphism of sheaves of abelian groups on a topological space X.

(1) Let $f\colon \mathscr{F} \to \mathscr{E}$ be a morphism of sheaves. For any $x \in X$, we define a map $f_x\colon \mathscr{F}_x \to \mathscr{E}_x$ as follows: For any $s \in \mathscr{F}(U)$ with U an open neighborhood of x, we define

$$f_x([s]) \coloneqq [f_U(s)] \in \mathscr{E}_x.$$

Prove that f_x is a well-defined homomorphism of abelian groups.

(2) Recall that a morphism $f\colon \mathscr{F} \to \mathscr{E}$ is called injective if and only if $f_x\colon \mathscr{F}_x \to \mathscr{E}_x$ is an injective homomorphism of abelian groups. Prove that f is injective if and only if $f_U\colon \mathscr{F}(U) \to \mathscr{E}(U)$ is injective for any open subset U of X.

(3) Let $X = \{z \in \mathbb{C} \mid |z| = 1\} = S^1$. For any open subset U of X, define

$$\mathscr{E}(U) \coloneqq \{s\colon U \to \mathbb{C} \mid s \text{ is continuous}\},$$

and

$$\mathscr{Q}(U) \coloneqq \{s\colon U \to \mathbb{C}^* \mid s \text{ is continuous}\}.$$

Define a morphism $g\colon \mathscr{E} \to \mathscr{Q}$ such that for any open subset U of X, we have

$$g_U\colon \mathscr{E}(U) \to \mathscr{Q}(U),\ s \mapsto \exp \circ s$$

Prove that g is surjective; in other words, the induced morphism $g_x\colon \mathscr{E}_x \to \mathscr{Q}_x$ is surjective for any $x \in X$.

(4) In (2) above, proved that $g_X\colon \mathscr{E}(X) \to \mathscr{Q}(X)$ is not surjective.

Exercise 2 (Cohomologies of the projective line)**.** Denote by $\bar{\mathbb{C}}$ the *extended complex plane*, i.e., $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

(1) Prove that $\mathbb{P}^1$ is homeomorphic to $\bar{\mathbb{C}}$.

(2) Prove that

$$H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k)) = \begin{cases} 0 & \text{if } k < 0, \\ \mathbb{C} & \text{if } k = 0, \\ \text{homogeneous polynomials of two variables of degree k} & \text{if } k \geq 1. \end{cases}$$

(3) Let $\mathcal{U} = \{U_0, U_1\}$ be the standard open covering of $\mathbb{P}^1$, i.e.

$$U_i = \{[x_0 : x_1] \in \mathbb{P}^1 \mid x_i \neq 0\}.$$

For any $k, q \in \mathbb{Z}$, compute that Cěch cohomology $\check{H}^q(\mathcal{U}, \mathscr{O}_{\mathbb{P}^1}(k))$.

(4) For any $k, p \in \mathbb{Z}$, compute the cohomology $H^q(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k))$.

(5) Prove that the holomorphic tangent bundle $T_{\mathbb{P}^1}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(2)$.

Exercise 3 (Fundamental class of divisors)**.** Let X be a compact complex manifold of dimension n and let D be an irreducible smooth hypersurface. Denote by $\iota\colon D \to X$ the natural inclusion.

(1) Show that the map

$$\mathcal{A}^{2n-2}_{\mathbb{C}}(X) \to \mathbb{C}, \quad \omega \mapsto \int_D \iota^*\omega$$

induces a linear form

$$[D]\colon H^{2n-2}(X, \mathbb{C}) \to \mathbb{C}.$$

By Poincaré duality $H^2(X,\mathbb{C}) \cong H^{2n-2}(X,\mathbb{C})^*$, so $[D]$ can be seen as a cohomology class in $H^2(X,\mathbb{C})$, called the *fundamental class* of D.

(2) (**very hard, See [Huybrechts,Proposition 4.4.13] or [Demailly, Theorem 13.4]**) Let $c_1(D) \in H^2(X,\mathbb{C})$ be the first Chern class of the holomorphic line bundle L_D. Show that $c_1(D) = [D]$.

Exercise 4 (Chern–Weil Theorem)**.** Let X be a differential manifold and let $E \to X$ be a complex vector bundle.

(1) Let ∇ and ∇' be two connection on E. Prove that $\nabla - \nabla' \in \mathcal{A}^1_{\mathbb{C}}(X, \mathrm{End}(E))$.

(2) Let $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$ and $\beta \in \mathcal{A}^{q'}_{\mathbb{C}}(X, \mathrm{End}(E))$. We define $[\alpha, \beta]$ as follows:

$$[\alpha, \beta] \coloneqq \alpha \wedge \beta - (-1)^{qq'} \beta \wedge \alpha,$$

which is called the *(super)-commutator*. Prove that $\mathrm{tr}[\alpha, \beta] = 0$.

(3) For a connection ∇ and $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$, we define

$$[\nabla, \alpha] \colon \mathcal{A}_{\mathbb{C}}(E) \to \mathcal{A}^{q+1}_{\mathbb{C}}(E), \quad s \mapsto \nabla(\alpha \wedge s) - (-1)^q \alpha \wedge \nabla(s),$$

where s is a local differential section of E. Prove that $[\nabla, \alpha] \in \mathcal{A}^{q+1}_{\mathbb{C}}(X, \mathrm{End}(E))$.

(4) (**Bianchi identity**) For any $k \in \mathbb{Z}_{>0}$, prove that $[\nabla, \Theta^k_\nabla] = 0$.

(5) For any $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$, prove that $\mathrm{tr}([\nabla, \alpha])$ is independent of choice of the connection ∇.

(6) For any $\alpha \in \mathcal{A}^q_{\mathbb{C}}(X, \mathrm{End}(E))$, prove that $\mathrm{d}(\mathrm{tr}(\alpha)) = \mathrm{tr}[\nabla, \alpha]$.

(7) Let $f(t) = a_0 + a_1 t + \cdots + a_m t^m \in \mathbb{C}[t]$ be a polynomial in one variable t. We will denote by $f(\Theta_\nabla)$ the formal sum

$$f(\Theta_\nabla) \coloneqq a_0 + a_1 \Theta_\nabla + \cdots + a_m \Theta^m_\nabla \in \mathcal{A}^\bullet_{\mathbb{C}}(X, \mathrm{End}(E)).$$

Prove that $\mathrm{d}\ \mathrm{tr}(f(\Theta_\nabla)) = 0$.

(8) Let ∇ and ∇' be two connections on E. Define $\nabla_t = t\nabla + (1-t)\nabla'$ for any $t \in [0,1]$. Prove that ∇_t is again a connection and

$$\frac{\mathrm{d}\Theta_{\nabla_t}}{\mathrm{d}t} = [\nabla_t, \frac{\mathrm{d}\nabla_t}{\mathrm{d}t}]$$

(9) For any polynomial $f(t) \in \mathbb{C}[t]$, prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\Theta_{\nabla_t}) = \mathrm{d}\ \mathrm{tr}\left(\frac{d\nabla_t}{dt} f'(\Theta_{\nabla_t})\right)$$

(10) Prove that there exists a differential form $\eta \in \mathcal{A}^\bullet_{\mathbb{C}}(X)$ such that

$$\mathrm{tr} f(\Theta_\nabla) - \mathrm{tr} f(\Theta_{\nabla'}) = \mathrm{d}\eta.$$

(11) Show that the Chern classes $c_i(E, \nabla)$ in terms of curvatures is well-defined and is independent of the choice of ∇.