Exercise sheet 1

Exercise 1 (Complex structure)**.** Let E be a real vector space and let J be a complex structure on E; that is, $J\colon E \to E$ is an endomorphism such that $J^2 = \mathrm{id}_E$.

(1) Prove that E admits a complex vector space structure such that the multiplication by i is given by J. Conversely, prove that all complex vector space, viewed as a real vector space, has a natural complex structure.

(2) Prove that the action $J \otimes_{\mathbb{R}} \mathbb{C}$ on $E \otimes_{\mathbb{R}} \mathbb{C}$ is diagonalizable, its eigenvalues are i and $-i$, the corresponding eigenspaces are:
$$E^{1,0} \coloneqq \{X - iJ(X) \mid X \in E\}, \quad E^{0,1} \coloneqq \{X + iJ(X) \mid X \in E\}.$$

(3) Prove that the map $E \to E^{1,0}$ by sending X to $X - iJ(X)$ is an isomorphism of complex vector spaces.

(4) Let (F, J_F) be another real vector space with a complex structure J_F and let $L\colon E \to F$ be a real linear map. Prove that L is a $\mathbb{C}$-linear map if and only if $LJ_E = J_F L$ if and only if $L(E^{1,0}) \subset F^{1,0}$ if and only if $L(E^{0,1}) \subset F^{0,1}$.

(5) Prove the the restriction $\mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ is an isomorphism.

(6) Prove that for any $\ell \in \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C})$, the restriction $\ell|_E$ is $\mathbb{C}$-linear if and only if $\ell(E^{0,1}) = 0$.

(7) Prove that the adjoint map J^t of J is a complex structure of E^* and $(E^*)^{1,0} = \ker(J^t - \mathrm{id}_{E^*})$ is identified to $\mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$.

(8) Denote $\wedge^{p,q} = (\wedge^p E^{1,0}) \otimes (\wedge^q E^{0,1})$. Prove that we have
$$\wedge^k (E \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p+q=k} \wedge^{p,q} E$$
and give a basis for these vector spaces.

Exercise 2 (Grassmann varieties and Plücker embedding)**.** Let $n \geq 2$ be a positive integer and let $0 < r < n$ be an integer. We define the Grassmann variety as the following set:
$$\mathrm{Gr}(r, \mathbb{C}^n) \coloneqq \{S \subset \mathbb{C}^n \text{ linear subspace of dimension } r\}.$$
Let $\mathrm{U}(n, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$ be the unitary group.

(1) Prove that there exists a surjective map $U(n, \mathbb{C}) \to \mathrm{Gr}(r, \mathbb{C}^n)$. Then we define $\mathrm{Gr}(n, \mathbb{C}^n)$ as a topological space with the induced quotient topology.

(2) We define an atlas on $\mathrm{Gr}(r, \mathbb{C}^n)$ as following: For any linear subspace $T_i \subset \mathbb{C}^n$ of dimension $n - r$, we let
$$U_i \coloneqq \{S \subset \mathbb{C}^n \text{ linear subspace of dimension } r \mid S \cap T_i = 0\}.$$
For any $S_i \in U_i$, we define a map
$$\phi_i \colon U_i \longrightarrow \mathrm{Hom}(S_i, T_i) \cong \mathbb{C}^{n(n-r)}$$
by sending $S \in U_i$ the unique $\mathbb{C}$-linear map $f \in \mathrm{Hom}(S_i, T_i)$ such that $S \subset \mathbb{C}^n = S_i \oplus T_i$ is the graph of f. Prove that the maps ϕ_i define a complex structure on $\mathrm{Gr}(r, \mathbb{C}^n)$.

(3) We define a map
$$\psi\colon \mathrm{Gr}(r, \mathbb{C}^n) \longrightarrow \mathbb{P}(\wedge^r \mathbb{C}^n)$$
as following: Let $S \subset \mathbb{C}^n$ be a linear subspace of dimension r and let $u_1, \ldots, u_r$ be a basis of U. The multi-vector $u_1 \wedge \cdots \wedge u_r$ gives a point in $\mathbb{P}(\wedge^r \mathbb{C}^n)$. Prove that ψ is well-defined and is an embedding (called the *Plücker embedding*).

(4) Prove that $\mathrm{Gr}(r, \mathbb{C}^n)$ is a projective manifold. (**Hints:** Prove that the image $\mathrm{im}(\psi)$ can be identified to the multi-vectors w in $\wedge^r \mathbb{C}^n$ which are decomposable; that is, there exist vectors $v_1, \ldots, v_r \in \mathbb{C}^n$ such that $w = v_1 \wedge \cdots \wedge v_r$. For any $w \in \wedge^r \mathbb{C}^n$, we consider the linear map

$$\phi_w \colon \mathbb{C}^n \longrightarrow \wedge^{r+1} \mathbb{C}^n, \quad v \mapsto v \wedge w.$$

Prove that w is decomposable if and only if $\mathrm{rank}(\phi_w) \leq n - r$.)

(5) Let $e_1, \ldots, e_4$ be a basis of $\mathbb{C}^4$. Then every 2-vector $w \in \wedge^2 \mathbb{C}^4$ has a unique decomposition

$$w = X_0 e_1 \wedge e_2 + X_1 e_1 \wedge e_3 + X_2 e_1 \wedge e_4 + X_3 e_2 \wedge e_3 + X_4 e_2 \wedge e_4 + X_5 e_3 \wedge e_4.$$

Prove that for the homogeneous coordinates $[X_0 : \cdots : X_5]$ over $\mathbb{P}(\wedge^2 \mathbb{C}^4)$, the Plücker embedding of $\mathrm{Gr}(2, \mathbb{C}^4)$ in $\mathbb{P}(\wedge^2 \mathbb{C}^4)$ is given by the equation

$$X_0 X_5 - X_1 X_4 + X_2 X_3 = 0.$$

Exercise 3 (Canonical bundles)**.** Let X be an n-dimensional complex manifold. Let S, E and Q be holomorphic vector bundles over X. Let $\phi \colon S \to E$ and $\psi \colon E \to Q$ be morphisms of vector bundles. We say that the sequence

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is *exact at* E if $\mathrm{im}(\phi) = \ker(\psi)$.

(1) Let $0 \to S \to E \to Q \to 0$ be an exact sequence; that is, a sequence of morphisms of vector bundles which are exact at S, E and Q. Prove that we have a canonical isomorphism

$$\det(E) \cong \det(S) \otimes \det(Q).$$

(2) Prove that over $\mathbb{P}^n$, we have the following exact sequence (called the *Euler sequence*):

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

and then derive that $\omega_{\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^n}(-n-1)$.

(3) Let $H \subset \mathbb{P}^n$ be a projective submanifold defined by a homogeneous polynomial of degree d. Prove that we have an exact sequence over H as follows:

$$0 \longrightarrow T_H \longrightarrow T_{\mathbb{P}^n}|_H \longrightarrow \mathscr{O}_{\mathbb{P}^n}(d)|_H \longrightarrow 0,$$

where $T_H \to T_{\mathbb{P}^n}|_H$ is the natural inclusion of tangent bundles. Deduce that

$$\omega_H \cong \mathscr{O}_{\mathbb{P}^n}(-n-1+d).$$

Generalize this result to smooth complete intersections in $\mathbb{P}^n$.

(4) Let $C \subset \mathbb{P}^3$ be the twisted cubic; that is, the image of map $\nu \colon \mathbb{P}^1 \to \mathbb{P}^3$ defined as follows:

$$[X_0 : X_1] \longmapsto [X_0^3 : X_0^2 X_1 : X_0 X_1^2 : X_1^3].$$

Prove that ν is an embedding and deduce that $C \subset \mathbb{P}^3$ is not a complete intersection.