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Fano manifolds such that the tangent bundle is (not) big

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Let X be a Fano manifold, i.e. a complex projective manifold such that the anticanonical bundle $-K_X = \det T_X$ is ample. One of the most basic questions one can ask about Fanos is whether the positivity of $\det T_X$ implies any positivity properties of T_X. Obviously, one should not expect too much: by Mori's theorem we know that the projective space is the only Fano manifold such that T_X is ample, and, more generally, one expects that semipositivity of the tangent bundle (e.g. metric conditions, T_X is nef, etc.) should only appear for very special geometries. In this project (which is joint work with Jie Liu and Feng Shao) we consider a much weaker condition:

Definition. *Let X be a projective manifold. We say that the tangent bundle is pseudoeffective (resp. big) if the tautological class $c_1(\mathscr{O}_{\mathbb{P}(T_X)}(1))$ on the projectivised bundle $\mathbb{P}(T_X)$ is pseudoeffective (resp. big).*

Examples of manifolds with pseudoeffective tangent bundle are almost homogeneous spaces (or more generally any manifold admitting a non-zero vector field); by a result Hsiao [6] all toric varieties have big tangent bundle. A priori, it is not clear if the property of having a big tangent bundle is very restrictive. In fact big vector bundles can be very pathological: given an ample line bundle $A \to X$ on any projective manifold, the vector bundle $V = A \oplus A^*$ is big, but its determinant is trivial and the restriction to any curve is not nef. More generally the Chern class inequalities that are so useful when dealing with nef vector bundles will not hold for pseudoeffective vector bundles.

As a first step towards a theory of Fano manifolds with big/pseudoeffective tangent bundles, we characterise this property for some of the standard examples:

Theorem 1. *For $n \geq 2$, let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then T_X is pseudoeffective if and only if X is a hyperplane or a quadric.*

For the proof we use the theory of Schur functors to compute explicitly the space of global sections of certain twists of the symmetric powers of the tangent bundle. This involves vanishing theorems of Brückmann and Rackwitz [2], Schneider [12] and Bogomolov-de Oliveira [1]. Since these theorems already fail for complete intersections of higher codimension, settling the question for del Pezzo surfaces of degree 4 and 5 requires a different approach: every Fano manifold carries a family of minimal rational curves, i.e. a family of rational curves $f : \mathbb{P}^1 \to X$ that dominates X and such that for a general point $x \in X$, the curves passing through x form a complete family. It is well-known that a general minimal rational curve is standard, i.e. one has

$$f^* T_X \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^p \oplus \mathscr{O}_{\mathbb{P}^1}^{n-p-1}$$

for some $p \in \{0, 1, \ldots, n-1\}$. By a famous theorem of Cho-Miyaoka-Shepherd Barron [4] and Kebekus [9] one has $p \neq n-1$ unless X is the projective space, so we can consider the curves $\tilde{l} \subset \mathbb{P}(T_X)$ corresponding to the trivial quotients

$$f^* T_X \to \mathscr{O}_{\mathbb{P}^1}.$$

Let $\check{\mathcal{C}} \subset \mathbb{P}(T_X)$ be the closure of the locus covered by these curves, then we call $\check{\mathcal{C}}$ the total dual VMRT of the family of minimal rational curves. This terminology is justified by the fact that for $x \in X$ general the fibre $\check{\mathcal{C}}_x$ is the projective dual of the VMRT $\mathcal{C}_x$ that plays a prominent rôle in the theory of Hwang and Mok [7]. The total dual VMRT is a divisor in $\mathbb{P}(T_X)$ unless the VMRT is dual defective, so we can consider that in “many” cases this construction yields a divisor. It turns out that the class of this divisor can be computed in a number of cases, leading to the following result:

Theorem 2. *Let S be a smooth del Pezzo surface of degree $d := (-K_S)^2$. Then one has*

$$T_S \textit{ is big if and only if } d \geq 5.$$

Let X be a 3-dimensional del Pezzo manifold, i.e. a smooth Fano threefold such that $-K_X = 2H$ where H is a Cartier divisor. Set $d := H^3$. Then one has

$$T_X \textit{ is big if and only if } d \geq 5.$$

In the surface case, related results were obtained by Paris [11], Hosono-Iwai-Matsumura [5] and Mallory [10].

If X is a del Pezzo threefold of degree d, a general element of the linear system $|H|$ is a del Pezzo surface S of degree d. Thus the two parts of Theorem 2 taken together imply that

$$(*) \qquad T_X \text{ is big if and only if } T_S \text{ is big.}$$

This property is very surprising: the tangent bundle T_S is a *subbundle* of $T_X|_S$, so we can not expect any relation between their positivity properties. For example the del Pezzo threefold of degree five is almost homogeneous, so it has many vector fields. However the del Pezzo surface of degree five has no vector fields, hence the inclusion

$$H^0(S, T_S) \hookrightarrow H^0(S, T_X|_S) \neq 0$$

is the zero map. The proof of our theorem gives a hint why the property $(*)$ holds: if one computes the class of the total dual VMRT of a family of lines on a del Pezzo threefold X, the result depends on the number of (-1)-curves in S ! While the details of the proof are somewhat technical, the general strategy is quite classical: we follow the computation of the invariants of the Fano variety of lines of the cubic threefold that appears in the seminal work of Clemens and Griffiths [3].

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A condition of existence of cscK metrics on spherical manifolds

Thibaut Delcroix

Consider a smooth compact Kähler manifold X. Determining if there exists a cscK metric or not in a given Kähler class is a very hard geometric analysis problem involving a fourth order highly non-linear PDE. The aim of the Yau-Tian-Donaldson conjecture is to translate this geometric analysis problem into an algebro-geometric problem, involving a condition of K-stability inspired by GIT stability and the Kobayashi-Hitchin correspondence for Hermite-Einstein metrics. It is not apparent in general that the K-stability condition is easier to check than the construction of cscK metrics. The resolution of the Yau-Tian-Donaldson in the special case of the anticanonical class however allowed different authors to solve the question of existence of Kähler-Einstein metrics on large classes of manifolds that seemed out of reach from analytic techniques, and we had the pleasure to present such a result [2] at Oberwolfach in 2017.

In the case of general polarizations, arguably the best result in the direction of the YTD conjecture was obtained by Donaldson in a series of papers starting in 2002 [3] and ending in 2009 [4]. There he showed the YTD conjecture holds for