PROJECTIVE MANIFOLDS CONTAINING AN AMPLE DIVISOR ISOMORPHIC TO A LOCALLY RIGID FANO COMPLETE INTERSECTION

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ABSTRACT. In this short note, we classify the following pair (X, A): X is a projective manifold and A is an ample divisor on X such that A is isomorphic to a smooth locally rigid Fano complete intersection in a rational homogeneous space S of Picard number one.

1. INTRODUCTION

It is well known from adjunction theory that most projective varieties cannot (except in trivial ways) be ample divisors and the varieties that can be ample divisors practically determine the varieties they are ample divisors on. In [Wat08], Watanabe classified projective manifolds containing an ample divisor isomorphic to a homogeneous space. In particular, the following theorem is proved.

1.1. Theorem.[Wat08, Theorem 1] *Let X be a projective manifold of dimension $n \geq 3$ containing an ample divisor A isomorphic to a rational homogeneous space. If $\rho(A) = 1$, then the pair $(X, \mathscr{O}_X(A))$ is isomorphic to one of the following:*

(1) $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$;
(2) $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(2))$ *and* $n \geq 4$;
(3) $(Q^n, \mathscr{O}_{Q^n}(1))$ *and* $n \geq 4$;
(4) $(\mathrm{Gr}(2, 2\mathrm{n}), \mathscr{O}(1))$ *and $n \geq 2$, where* $\mathrm{Gr}(2, 2\mathrm{n})$ *is the Grassmannian of* 2*-dimensional subspaces in an $2n$-dimensional vector space and $\mathscr{O}(1)$ is the ample generator of the Picard group of* $\mathrm{Gr}(2, 2\mathrm{n})$;
(5) $(E_6/P_1, \mathscr{O}(1))$, *where E_6/P_1 is the* 27*-dimensional rational homogeneous space of type E_6 and $\mathscr{O}(1)$ is the ample generator of the Picard group of E_6/P_1.*

Recall that a projective manifold X is called *locally rigid* if for any smooth deformation $\mathcal{X} \to \Delta$ with $\mathcal{X}_0 \simeq X$, we have $\mathcal{X}_t \simeq X$ for t in a small (analytic) neighborhood of 0. If X is a Fano manifold, by Akizuki-Nakano vanishing theorem, we have $h^q(X, T_X) = 0$ for all $q \geq 2$. Then, by Kodaira-Spencer's deformation theory, X is locally rigid if and only if $h^1(X, T_X) = 0$. In particular, the well known works of Bott show that all rational homogeneous spaces are locally rigid. Moreover, recently Bai, Fu and Manivel classified all locally rigid Fano complete intersection in rational homogeneous spaces with Picard number one.

1.2. Theorem.[BFM20, Theorem 1.1] *Let S be a rational homogeneous space of Picard number one and let $A \subset S$ be a smooth Fano complete intersection. Then A is locally rigid if and only if A is isomorphic to one of the following:*

Date: January 5, 2021.
2010 *Mathematics Subject Classification.* 14J45, 14M17, 14E30, 14N05, 14N30.
Key words and phrases. Fano manifolds, rational homogeneous spaces, locally rigidity, ample divisors, extremal contraction.

(1) $\mathbb{P}^n$ *or* Q^n*;*
(2) *a general hyperplane section of the following:*

$$\mathrm{Gr}(2,\mathrm{n})(\mathrm{n}\geq 5), \mathrm{Gr}(3,6), \mathrm{Gr}(3,7), \mathrm{Gr}(3,8), \mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_7,$$
$$\mathrm{Gr}_\omega(2,6), \mathrm{Lag}(3,6), \mathrm{F}_4/\mathrm{P}_4, \mathrm{E}_6/\mathrm{P}_1, \mathrm{E}_7/\mathrm{P}_7.$$

(3) *a general codimension* 2 *linear section of* $\mathrm{Gr}(2, 2\mathrm{n}+1)$*,* $n \geq 2$*;*
(4) *a general codimension* 2 *or* 3 *linear section of* $\mathbb{S}_5$*;*
(5) *a general codimension* 3 *or* 4 *linear section of* $\mathrm{Gr}(2,5)$*.*

Here $\mathrm{Gr}(\mathrm{k},\mathrm{n})$ is the Grassmannian of k-dimensional subspaces in a vector space of dimension n. $\mathbb{S}_n$ is the spinor variety, parameterizing n-dimensional isotropic linear subspaces in an orthogonal vector space of dimension $2n$. $\mathrm{Gr}_\omega(2,6)$ is the symplectic Grassmannian and $\mathrm{Lag}(3,6)$ is the Lagrangian Grassmannian, which parameterize, respectively, isotropic planes and Lagrangian subspaces in a 6-dimensional symplectic vector space. For simple Lie group G, we denote by P_i the maximal parabolic subgroup of G corresponding to the i-th root, where we use Bourbaki's enumeration of simple roots.

The main result is to classify those projective manifolds containing an ample divisor A isomorphic to one of the Fano manifolds in Theorem 1.2 and it can be viewed as a generalization of Theorem 1.1.

1.3. Theorem. *Let* X *be a projective manifold of dimension* $(n+1)$ *containing a smooth locally rigid ample divisor* A *with* $\rho(A) = 1$*, which is isomorphic to a Fano complete intersection in a rational homogeneous space* S *of Picard number one. If* $n \geq 2$*, then one of the following holds.*

(1) *A is isomorphic to a codimension* k *linear section of* S *and* X *is isomorphic to a codimension* $(k-1)$ *linear section of* S*.*
(2) *A is isomorphic to a quadric hypersurface and the pair* $(X, \mathscr{O}_X(A))$ *is isomorphic to either* $(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(2))$ *or* $(Q^{n+1}, \mathscr{O}_{Q^{n+1}}(1))$*.*

Acknowledgements. I would like to thank Baohua Fu for bringing me attention to this question and for patiently answering my numerous questions. For many useful comments on the draft of this paper I would like to express my deep gratitude. This work is supported by China Postdoctoral Science Foundation (2019M650873).

Notation and convention

Throughout we work over the field of complex numbers. If $\mathscr{E}$ is a vector bundle over a projective variety X, we denote by $\mathbb{P}(\mathscr{E})$ the Grothendieck projectivization and by $\mathscr{E}^*$ the dual sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$. If $\mathscr{E}$ is the vector bundle $\mathscr{O}_{\mathbb{P}^1}(a_0) \oplus \cdots \mathscr{O}_{\mathbb{P}^1}(a_d)$ with $a_0 \geq \cdots \geq a_d > 0$, we will denote by $S(a_0, \cdots, a_d)$ the embedding of the rational normal scroll $\mathbb{P}(\mathscr{E}) \subset \mathbb{P}^{\sum a_i + d}$ given by the tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

We denote by S a rational homogeneous space of Picard number one. In particular, the ample generator of $\mathrm{Pic}(S)$ will be denoted by $\mathscr{O}_S(1)$. If $A \subset S$ is a projective subvariety, we denote by $\mathscr{O}_A(1)$ the restriction $\mathscr{O}_S(1)|_A$.

Throughout this paper, we say that a projective manifold A satisfies (♣) if A satisfies the following two conditions:

- A is isomorphic to a locally rigid Fano complete intersection in a rational homogeneous space S of Picard number one

- A is isomorphic to neither a projective space nor a smooth quadric hypersurface.

According to Theorem 1.2, a projective manifold A satisfies (♣) if and only if it is isomorphic to one of the projective manifolds listed in (2)-(5) in Theorem 1.2.

2. Projective extension

Recall that a line bundle $\mathscr{L}$ over a projective variety X is called *simply generated* if the graded algebra

$$R(X, \mathscr{L}) := \bigoplus_{m=0} H^0(X, \mathscr{L}^{\otimes m})$$

is generated by $H^0(X, \mathscr{L})$ as a $\mathbb{C}$-algebra.

2.1. Proposition. [Liu20, Proposition 2.10 and 2.11] *Let X be a normal projective variety of dimension $n \geq 3$ and let $\mathscr{L}$ be an ample line bundle over X. Assume that $h^1(X, \mathscr{O}_X) = 0$ and $D \in |\mathscr{L}|$ is a member which is irreducible and reduced as a subscheme of X. Then the following statements hold.*

(1) If $\mathscr{L}|_D$ is simply generated, then $\mathscr{L}$ is very ample.

(2) Assume moreover that D is smooth, $h^1(D, T_D \otimes (\mathscr{L}^*|_D)) = 0$ and $\mathscr{L}|_D$ is simply generated. Then one of the following statements holds.

(2.1) The map $\Phi_{|\mathscr{L}|}$ defined by the complete linear system $|\mathscr{L}|$ sends X to be cone over $\Phi(Y)$.

(2.2) The pair $(D, \mathscr{L}|_D)$ is isomorphic to $(Q^{n-1}, \mathscr{O}_{Q^{n-1}}(1))$, where Q^{n-1} is a quadric hypersurface of dimension $n-1$.

2.2. Remark. In [Liu20, Proposition 2.10 and Proposition 2.11], the results above are stated under the hypothesis that $\mathscr{L}|_D$ is very ample. Nevertheless, as pointed out by Jinhyung Park, this is not correct since a very ample line bundle is not necessarily simply generated. However, we note that in the proof the very ampleness of $\mathscr{L}|_D$ is only used to derive that $\mathscr{L}|_D$ is simply generated.

To apply the result above to our situation, we need the following simple observation.

2.3. Lemma. *Let S be a rational homogeneous space of Picard number one and let $A \subset S$ be a smooth complete intersection. Then for any positive integer k, then the ample line bundle $\mathscr{O}_A(k)$ is simply generated.*

Proof. By [RR85, Theorem 1 (iii)], the ample line bundle $\mathscr{O}_S(k)$ is simply generated. Thus, to prove the lemma, it suffices to show that the natural restriction map

$$\rho_r \colon H^0(S, \mathscr{O}_S(r)) \to H^0(X, \mathscr{O}_X(r)) \tag{2.1}$$

is surjective for all integers $r \geq 1$. Without loss of generality, we may assume that A is a complete intersection of multi-degree $(d_1, \ldots, d_m)$ and denote by $E \to S$ the rank m vector bundle

$$\bigoplus_{i=1}^{m} \mathscr{O}_S(d_i).$$

Then we the following twisted Koszul complex

$$0 \to (\wedge^m E) \otimes \mathscr{O}_S(r) \to (\wedge^{m-1} E) \otimes \mathscr{O}_S(r) \to \cdots \to E(r) \to \mathscr{O}_S(r) \to \mathscr{O}_A(r) \to 0.$$

Then a standard spectral sequence argument shows that it is enough to show

$$H^i(S, (\wedge^i E) \otimes \mathscr{O}_S(r)) = 0$$

for any integer $i \geq 1$. This is well-known since $(\wedge^i E) \otimes \mathscr{O}_S(r)$ is a direct sum of ample line bundles and S is a rational homogeneous space of Picard number one (see for instance [RR85, Theorem 1 (i)]). □

3. VMRT AND RATIONAL HOMOGENEOUS SPACES

3.A. **Hilbert scheme of lines and VMRTs.** Let $X \subset \mathbb{P}^N$ be a non-degenerate projective manifold of dimension $n \geq 1$. Let $\mathcal{L}_{x,X}$ denote the Hilbert scheme of lines contained in X passing through the point $x \in X$. We define the morphism

$$\tau_x \colon \mathcal{L}_{x,X} \to \mathbb{P}(T^*_{x,X}) = \mathbb{P}^{n-1}$$

which associates to each line $[\ell] \in \mathcal{L}_{x,X}$ the corresponding tangent direction through x, i.e. $\tau_x([\ell]) = \mathbb{P}(T^*_{x,\ell})$. Then τ_x is a closed immersion. For $x \in X$ such that $\mathcal{L}_{x,X} \neq \emptyset$, we shall always identify $\mathcal{L}_{x,X}$ with $\tau_x(\mathcal{L}_{x,X})$ and we shall naturally consider $\mathcal{L}_{x,X}$ as a subscheme of $\mathbb{P}^{n-1} = \mathbb{P}(T^*_{x,X})$. We refer the reader to [Rus12] and the references therein for more details.

Recall that a *prime Fano manifold* is a Fano manifold of Picard number one so that the ample generator of the Picard group is very ample. The following result is certainly well known to experts. We include a proof for lack of explicit references.

3.1. Lemma. *Let X be an n-dimensional prime Fano manifolds of index $\geq (n+1)/2$, then X is ruled by lines. In particular, if $X \subset \mathbb{P}^N$ is the embedding of X given by the ample generator of* Pic(X)*, then the Hilbert scheme of lines $\mathcal{L}_{x,X} \subset \mathbb{P}(T^*_{x,X})$ at a general point x is smooth.*

Proof. Let l_X be the minimal anticanonical degree of a locally unsplit dominating family of rational curves in X. Then X is ruled by lines if and only if l_X equals to the index of X. If X is of index $> (n+1)/2$, the existence of lines follows from Mori's "bend-and-break" lemma (see [KM98, Theorem 1.10]). If X is of index $(n+1)/2$, then X is ruled by lines unless $l_X \geq n+1$. Since X is a Fano manifold, X is rationally connected. Thus, for a very general point $x \in X$, every rational curve passing through x is free. In particular, every rational curve passing through x is contained in a covering family of rational curves of X. If $l_X \geq n+1$, then for every rational curve C passing through x, we have $-K_X \cdot C \geq n+1$. As a consequence, X is isomorphic to $\mathbb{P}^n$ according to [CMSB02, Corollary 0.4 (11)]. In particular, X is of index $n+1$, a contradiction. Hence, X is ruled by lines. Furthermore, for prime Fano manifolds covered by lines, the smoothness of the Hilbert scheme of lines $\mathcal{L}_x$ for a general point x follows from [Rus12, Proposition 2.1]. □

3.2. Proposition.[Rus12, Proposition 3.2] *Let $X' \subset \mathbb{P}^{N+1}$ be an n-dimensional irreducible projective variety $(n \geq 2)$ which is a projective extension of the non-degenerate projective variety $X \subset \mathbb{P}^N$. Let $x \in X$ be an arbitrary point such that $\mathcal{L}_{x,X} \neq \emptyset$. Then*

(1) $\mathcal{L}_{x,X'} \cap \mathbb{P}(T^*_{x,X}) = \mathcal{L}_{x,X}$ as schemes.
(2) If $x \in X$ is general, then $\dim_{[\ell]}(\mathcal{L}_{x,X'}) = \dim_{[\ell]}(\mathcal{L}_{x,X}) + 1$ and $[\ell]$ is a smooth point of $\mathcal{L}_{x,X'}$ for every $[\ell] \in \mathcal{L}_{x,X}$.

Let X be a uniruled projective manifold. We choose a family $\mathcal{K}$ of minimal rational curves, i.e. an irreducible component of the space of rational curves on X such that for a general point $x \in X$, the subscheme $\mathcal{K}_x \subset \mathcal{K}$ parametrizing members of $\mathcal{K}$ passing through x is nonempty and projective. Then the *tangent map* at x is the rational map $\mathcal{K}_x \dashrightarrow \mathbb{P}(T^*_{x,X})$ which sends a member of $\mathcal{K}_x$ smooth at x to its tangent direction at x. Let $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ be the strict image of the tangent map. Then $\mathcal{C}_x$ is called the *variety of minimal rational tangents* (VMRT) of X at x. For a general point x, we know that $\mathcal{K}_x$ is smooth and the tangent map is the normalization of $\mathcal{C}_x$.

In particular, if $X \subset \mathbb{P}^N$ is a non-degenerate projective manifold ruled by lines, we fix some dominant irreducible component, say $\mathcal{K}$, of the Hilbert scheme of lines of X. Then for a general point $x \in X$, we can define the VMRT $\mathcal{C}_x$ associated to $\mathcal{K}$ so that $\mathcal{C}_x \subseteq \mathcal{L}_{x,X} \subset \mathbb{P}^{n-1}$. Thus, if $\mathcal{L}_{x,X}$ is irreducible, then $\mathcal{C}_x = \mathcal{L}_{x,X}$ and X has only one maximal irreducible covering family of lines.

3.B. **Rational homogeneous spaces and odd symplectic Grassmannians.** The knowledge of the VMRTs of rational homogeneous spaces of Picard number one is particularly important, since rational homogeneous spaces are determined by $\mathcal{C}_x$ and its embedding in $\mathbb{P}(T^*_{x,X})$ within the class of Fano manifolds of Picard number one. In fact, in the very recent work of [HL19], it turns out that odd symplectic Grassmannians, in the sense of [Mih07], can be also characterized via VMRTs.

Let us briefly review the basic facts of odd symplectic Grassmannians. Let V be a complex vector space endowed with a skew-symmetric bilinear form ω with maximal rank. We denote the variety of all k-dimensional isotropic subspaces of V by

$$\mathrm{Gr}_\omega(k,V) := \{W \subset V \mid \dim W = k, \omega|_W \equiv 0\}.$$

When $\dim(V)$ is even, say $2n$, the form ω is a non-degenerate symplectic form and this variety $\mathrm{Gr}_\omega(k,2n)$ is the usual symplectic Grassmannian, which is homogeneous under the symplectic group Sp(2n). However, when $\dim(V)$ is odd, say $2n+1$, the skew-form ω has the one-dimensional kernel $\ker(\omega)$. Then variety $\mathrm{Gr}_\omega(k,2n+1)$, called the *odd symplectic Grassmannian*, is not homogeneous and has two orbits under the action of its automorphism group if $2 \leq k \leq n$.

3.3. Theorem.[Mok08, HH08, HL19, HLT19] *Let S be a rational homogeneous space of Picard number one or an odd symplectic Grassmannian. Let X be a Fano manifold of Picard number one with a family $\mathcal{K}$ of minimal rational curves. Suppose that the VMRT $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ of $\mathcal{K}$ at a general point $x \in X$ is projectively equivalent to the VMRT $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ of S at a general point $s \in S$. Then X is isomorphic to S.*

3.4. Remark. The theorem above was proved for rational homogeneous spaces with Picard number one associated to a long root in [Mok08] and [HH08]. Very recently, Hwang and Li proved it for symplectic Grassmannians and odd symplectic Grassmannians in [HL19]. For the remaining cases, F_4/P_3 and F_4/P_4, it is proved in the upcoming paper of Hwang-Li-Timashev [HLT19].

For the rational homogeneous spaces considered in this paper, we collect the VMRT and its embedding in the following table and we refer the reader to [LM03] for more details.

$\mathcal{D}$	node	G/P	dimension	index	VMRT	embedding

A_n	r	$\mathrm{Gr(r, n+1)}$	$r(n+1-r)$	$n+1$	$\mathbb{P}^{r-1} \times \mathbb{P}^{n-r}$	$\mathcal{O}(1,1)$
C_n	n	$\mathrm{Lag(n, 2n)}$	$n(n+1)/2$	$n+1$	$\mathbb{P}^{n-1}$	$\mathcal{O}(2)$
D_n	n	$\mathbb{S}_n$	$n(n-1)/2$	$2n-2$	$\mathrm{Gr(2, n)}$	$\mathcal{O}(1)$
F_4	P_4	F_4/P_4	15	11	smooth hyperplane section of $\mathbb{S}_5$	$\mathcal{O}(1)$
E_6	P_1	E_6/P_1	16	12	$\mathbb{S}_5$	$\mathcal{O}(1)$
E_7	P_7	E_7/P_7	27	18	E_6/P_1	$\mathcal{O}(1)$

We also need the following description of the VMRTs of odd symplectic Grassmannians.

3.5. Proposition. *The odd symplectic Grassmannian $S{:=}\mathrm{Gr}_\omega(k, 2n+1)$ $(2 \leq k \leq n)$ is a Fano manifold with $\rho(S) = 1$ and, as sets, it is a linear section of $\mathrm{Gr(k, 2n+1)}$. Moreover, if $s \in S$ is a general point, then the VMRT $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ of S at s is projectively equivalent to the following projective bundle embedded by the tautological bundle*

$$\mathbb{P}(\mathscr{O}_{\mathbb{P}^{k-1}}(2) \oplus \mathscr{O}_{\mathbb{P}^{k-1}}(1)^{\oplus(2n+1-2k)}) \subset \mathbb{P}^{(2n+1-2k)k+k(k+1)/2-1}.$$

Proof. See [Mih07, Proposition 4.1 and Proposition 5.6] and [Par16, Proposition 2.1 and Lemma 3.1]. ☐

3.6. Remark. Let S be a rational homogeneous space of Picard number one, and let $\mathscr{L}$ be the ample generator of $\mathrm{Pic}(S)$. Then $|\mathscr{L}|$ defines the unique minimal G-equivariant embedding $S \subset \mathbb{P}(V_\lambda)$ corresponding to the highest weight. Then the VMRT $\mathcal{C}_o \subset \mathbb{P}(T^*_{s,S})$ of S at a referenced point $s \in S$ coincides with the Hilbert scheme of lines $\mathcal{L}_{s,S} \subset \mathbb{P}(T^*_{s,S})$ of $S \subset \mathbb{P}V_\lambda$ at s (see for instance [LM03, Theorem 4.3 and 4.8]). Furthermore, the minimal rational curves of the odd symplectic Grassmannian $\mathrm{Gr}_\omega(k, 2n+1)$ are lines of the Grassmannian $\mathrm{Gr(k, 2n+1)}$ contained in $\mathrm{Gr}_\omega(k, 2n+1)$. Thus the VMRT $\mathcal{C}_s$ of $\mathrm{Gr}_\omega(k, 2n+1)$ at a general point s also coincides with the Hilbert scheme of lines of $\mathrm{Gr}_\omega(k, 2n+1) \subset \mathbb{P}^{4n-3}$ at s, where the embedding is given by the ample generator of its Picard group (see [Par16, Lemma 3.1]).

4. Locally rigid Fano manifolds as ample divisors

The aim of this section is to prove Theorem 1.3. Let X be a projective manifold of dimension $n+1 \geq 3$ containing a smooth ample divisor A which is isomorphic to a locally rigid Fano complete intersection in a rational homogeneous space S of Picard number one. If A is isomorphic to $\mathbb{P}^n$ or Q^n, then such pairs (X, A) have been already classified (see Theorem 1.1). In particular, it is easy to see that Theorem 1.3 holds in these two cases. Thus, we shall make the following assumption throughout this section:

(♣) the divisor A is not isomorphic to $\mathbb{P}^n$ nor Q^n.

As a consequence, the rational homogeneous space S is not isomorphic to a projective space nor a quadric hypersurface (see [**?**, Proposition 2.13]). In particular, thanks to Kobayashi-Ochiai's theorem, the index r of S is at most $\dim(S) - 1$.

4.A. **Degree of $\mathscr{O}_X(A)$ in $\mathrm{Pic}(X)$.** In the following proposition, we collect several properties of locally rigid smooth Fano complete intersections in rational homogeneous spaces of Picard number one.

4.1. Proposition. *Let S be a rational homogeneous space of Picard number one, and let A be a smooth locally rigid Fano complete intersection in S. Denote by n the dimension of A and by r the index of A. If A is not isomorphic to a codimension 4 linear section of $\mathrm{Gr}(2,5)$, or equivalently if $\rho(A)=1$, then the following statements hold.*

(1) $n \geq 3$ with equality if and only if A is a codimension 3 linear section of $\mathrm{Gr}(2,5)$. In particular, the restriction map $\mathrm{Pic}(S) \to \mathrm{Pic}(A)$ is an isomorphism.
(2) $r \geq n/2$ with equality if and only if A is a hyperplane section of $\mathrm{Gr}(3,8)$.
(3) $r \geq 3$ unless A is a codimension 3 linear section of $\mathrm{Gr}(2,5)$.

Proof. By assumption (♣), we are in cases (2)-(5) of Theorem 1.2. Recall that the symplectic Grassmannian $\mathrm{Gr}_\omega(k,2n)$ has index $2n+1-k$ and dimension $2k(n-k)+k(k+1)/2$. Then the result follows directly from Theorem 1.2 and the table given in the previous section. □

4.2. Remark. If A is isomorphic to a codimension 4 linear section of $\mathrm{Gr}(2,5)$, then A is a del Pezzo surface of degree 5 and $\rho(A)=5$.

The following useful lemma says that X is actually a prime Fano manifold and $\mathscr{O}_X(A)$ is the ample generator of $\mathrm{Pic}(X)$.

4.3. Proposition. *Let X be a projective manifold containing a smooth locally rigid ample divisor A with $\rho(A)=1$, which is isomorphic to a codimension k Fano complete intersection in a rational homogeneous space S of Picard number one. Then X is a prime Fano manifold such that $\mathrm{Pic}(X) \simeq \mathbb{Z}\mathscr{O}_X(A)$.*

Proof. Set $n = dim(A)$. Then we have $n \geq 3$ and the Lefschetz hyperplane theorem says that the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(A)$ is an isomorphism. Denote by $\mathscr{O}_X(1)$ the ample generator of $\mathrm{Pic}(X)$. Then $\mathscr{O}_A(1) := \mathscr{O}_X(1)|_A$ is the ample generator of $\mathrm{Pic}(A)$. By adjunction formula, it is easy to see that X is a Fano manifold because A is a Fano manifold. In particular, we have $h^1(X,\mathscr{O}_X)=0$. Then, according to Proposition **??**, $\mathscr{O}_X(A)$ is very ample. Let d be the positive integer such that $\mathcal{O}_X(A) \simeq \mathcal{O}_X(d)$. By assumption (♣), the pair $(A, \mathscr{O}_A(d))$ is not isomorphic to the pair $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$ nor the pair $(Q^n, \mathscr{O}_{Q^n}(1))$. As X is smooth, by Proposition **??**, we get $h^1(A, T_A(-d)) \neq 0$.

Let r be the index of A. Thanks to Proposition 4.1, we have $r \geq n/2$ with equality if and only if A is a hyperplane section of $\mathrm{Gr}(3,8)$. On the other hand, by adjunction formula, the index of X is $r+d$. Under the assumption (♣), X and S are not isomorphic to $\mathbb{P}^{n+1}$ nor Q^{n+1} (see [**?**, Proposition 2.13]). In particular, we have $r+d \leq \dim(X)-1=n$. It follows that $d \leq n/2$ with equality only if A is a hyperplane section of $\mathrm{Gr}(3,8)$. As a consequence, we get $r-d \geq 0$ with equality only if A is a hyperplane section of $\mathrm{Gr}(3,8)$.

Finally we show that $h^1(A, T_A(-d)) \neq 0$ if and only if $d=1$. Consider the following twisted normal sequence of A in S

$$0 \to T_A(-d) \to T_S(-d)|_A \to \mathscr{N}_{A/S}(-d) \to 0.$$

Then it follows that if $h^1(A, T_A(-d)) \neq 0$, we have either $h^1(A, T_S(-d)|_A) \neq 0$ or $h^0(A, \mathscr{N}_{A/S}(-d)) \neq 0$. In the latter case, since A is a linear section of S, we get

$d = 1$. Thus it remains to consider the former case. Since A is not isomorphic to a quadric hypersurface, by Theorem 1.2, A is always a linear section of S. Thus, by adjunction formula and Lefschetz hyperplane theorem, the index of S is equal to $r + k$. Set $\mathscr{E} = \mathscr{O}_S(-1)^{\oplus k}$, where $\mathscr{O}_S(1)$ is the ample generator of $\mathrm{Pic}(S)$. Then we have the following Koszul complex

$$0 \to \wedge^k \mathscr{E} \to \cdots \to \mathscr{E} \to \mathscr{O}_S \to \mathscr{O}_A \to 0.$$

Tensoring it with $T_S(-d)$ yields

$$0 \to \wedge^k \mathscr{E} \otimes T_S(-d) \to \cdots \to \mathscr{E} \otimes T_S(-d) \to T_S(-d) \to T_S(-d)|_A \to 0.$$

Then $h^1(A, T_S(-d)|_A) \neq 0$ only if there exists $0 \leq i \leq k$ such that

$$h^{n+k-i-1}(S, \wedge^i \mathscr{E}^* \otimes \Omega_S(-r-k+d)) = h^{i+1}(S, \wedge^i \mathscr{E} \otimes T_S(-d)) \neq 0.$$

If $r > d$ or $i < k$, then $\wedge^i \mathscr{E}^* \otimes \mathscr{O}_S(-r-k+d)$ is negative and consequently, by Akizuki-Nakano vanishing theorem, for $i > 0$, we have

$$h^{i+1}(S, \wedge^i \mathscr{E} \otimes T_S(-d)) = 0.$$

If $r = d$ and $i = k$, then S is isomorphic to $\mathrm{Gr}(3, 8)$ and A is a hyperplane section. In particular, in this case we have

$$h^{i+1}(S, \wedge^i \mathscr{E} \otimes T_S(-d)) = h^2(S, \mathscr{O}_S(-1) \otimes T_S(-d)) = h^2(S, \Omega_S).$$

It is well known that, for an irreducible Hermitian symmetric space S of compact type, $h^i(S, \Omega_S) \neq 0$ if and only if $i = 1$. Therefore, for $S \simeq \mathrm{Gr}(3, 8)$ and $i > 0$, we still have $h^{i+1}(S, \wedge^i \mathscr{E} \otimes T_S(-d)) = 0$. As a consequence, if $h^1(S, T_S(-d)|_A) \neq 0$, then we have $h^1(S, T_S(-d)) \neq 0$, which is possible only if $d = 1$ because of the assumption (♣) (see [MS99, Theorem B]). □

As an immediate application of Proposition 4.3, we prove Theorem 1.3 in several special cases.

4.4. Theorem. *Let X be a projective manifold containing a smooth locally rigid ample divisor A with $\rho(A) = 1$. Suppose that A is isomorphic to a codimension k linear section of a rational homogeneous space S which is one of the following:*

$$\mathrm{Gr}_\omega(2, 6), \mathrm{Lag}(3, 6), \mathbb{S}_5, \mathrm{Gr}(2, 5).$$

Then X is a locally rigid codimension $(k-1)$ linear section of S.

Proof. Denote by r the index of A, by n the dimension A and by $d{:=}\mathscr{O}_A(1)^n$ the degree of A. Let us denote d_X (resp. d_S) of X (resp. S) the positive integer $\mathscr{O}_X(A)^{n+1}$ (resp. $\mathscr{O}_S(A)^{n+1}$). As $\rho(A) = 1$, by Proposition 4.1, we have $n \geq 3$. Thus the restrictions $\mathrm{Pic}(X) \to \mathrm{Pic}(A)$ and $\mathrm{Pic}(S) \to \mathrm{Pic}(A)$ are both isomorphisms. Thanks to Proposition 4.3, we obtain $d_X = d = d_S$. On the other hand, by adjunction formula, the index of X is equal to the index $r + 1$.

If S is isomorphic $\mathrm{Gr}(2, 5)$, then A is a degree 5 del Pezzo manifold of dimension $6 - k$. It follows that X is a degree 5 del Pezzo manifold of dimension $7 - k$. By the classification of del Pezzo manifolds (see [Fuj90, Chapter I, Theorem 8.11]), X is isomorphic to a codimension $(k-1)$ linear section of $\mathrm{Gr}(2, 5)$. To see the locally rigidity of X, note that, up to projective equivalence, there is only one class of codimension k smooth linear section of $\mathrm{Gr}(2, 5)$ for $1 \leq k \leq 4$.

If S is isomorphic to $\mathrm{Gr}_\omega(2, 6)$ (resp. $\mathrm{Lag}(3, 6)$), then A is a hyperplane section of S and A is a degree 14 (resp. 16) Mukai manifold of dimension 6 (resp. 5). It follows that X is a degree 14 (resp. 16) Mukai manifold of dimension 7 (resp. 6). By the classification of Mukai manifolds given in [Muk89], X is isomorphic to S and the locally rigidity of X is clear.

Suppose now that S is isomorphic to the 10-dimensional spinor variety $\mathbb{S}_5$. Since the restriction map $H^0(S, \mathscr{O}_S(1)) \to H^0(A, \mathscr{O}_A(1))$ is surjective, the embedding $A \subset \mathbb{P}^{15-k}$ given by the complete linear system $|\mathscr{O}_A(1)|$ is projectively equivalent to a codimension k linear section of the minimal embedding $\mathbb{S}_5 \subset \mathbb{P}^{15}$. Moreover, since A is locally rigid, without loss of generality, we may assume that A is cut out by a general codimension k linear subspaces of $\mathbb{S}_5 \subset \mathbb{P}^{15}$. Note that $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is a self-dual variety and $k \leq 3$, then Proposition **??** implies that

$$\mathrm{def(A) = def(\mathbb{S}_5) - k = 4 - k.}$$

On the other hand, consider the embedding $X \subset \mathbb{P}^N$ defined by $\mathscr{O}_X(A) \simeq \mathscr{O}_X(1)$. Since $\mathscr{O}_X(A)|_A$ is isomorphic to $\mathscr{O}_A(1)$ and the restriction map $H^0(X, \mathscr{O}_X(1)) \to H^0(A, \mathscr{O}_A(1))$ is surjective, we get $N = 16 - k$ and the hyperplane section $A \subset \mathbb{P}^{15-k}$ of $X \subset \mathbb{P}^{16-k}$ is again projectively equivalent to the embedding of A given by $\mathscr{O}_A(1)$. Thus, by Proposition **??**, we obtain

$$\mathrm{def(X) = def(A) + 1 = 5 - k.}$$

It follows that $\dim(X) = \dim(X^*) + (k-1)$, where X^* is the projective dual variety of $X \subset \mathbb{P}^{16-k}$. As $k \leq 3$, by [Ein86, Mn97, Mn99], one checks directly that X is isomorphic to a codimension $(k-1)$ linear section of $\mathbb{S}_5$. □

4.5. Remark. Our argument in the case $S = \mathbb{S}_5$ can be also applied to the case $S = Gr(2,5)$ and $k \leq 2$ because the Plüker embedding $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ is also a self-dual variety. However, if $S = \mathrm{Gr}(2,5)$ and $k = 3$, then we cannot apply Proposition **??** to obtain $\mathrm{def(X) = def(A) + 1}$ because we may have $\mathrm{def(X)} = 0$ as $\mathrm{def(A)} = 0$.

4.6. Example. Let $\mathbb{S}_5 \subset \mathbb{P}^{15}$ be the 10-dimensional spinor variety, and let $X \subset \mathbb{S}_5$ be a smooth codimension 2 linear section. Recall that X is said to be *special* if X contains a 4-dimensional linear space $\mathbb{P}^4$, or equivalently there exists a line $l \subset X$ such that

$$N_{l/X} \cong \mathscr{O}_{\mathbb{P}^1}(-2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 6}.$$

According to [Kuz18, Corollary 6.9], there are exactly two isomorphism classes of smooth codimension 2 linear section of $\mathbb{S}_5$, the special one and the non-special one.

Denote by $\mathbb{G}$ the Grassmaniann of lines $\mathrm{Gr}(\mathbb{P}^1, \check{\mathbb{P}}^{15})$, where $\check{\mathbb{P}}^{15}$ is the dual space of $\mathbb{P}^{15}$. Let $K \in \mathbb{G}$ be a point. Then K corresponds a codimension 2 linear subspace L_K of $\mathbb{P}^{15}$ and denote by X_K the intersection $X \cap L_K$. Then there exists a quadratic divisor $R \in |\mathscr{O}_{\mathbb{G}}(2)|$, the *spinor quadratic line complex*, such that R contains a dense Zariski open subset R_0 which parametrising the smooth special codimension 2 linear section of $\mathbb{S}_5$.

Let A be a locally rigid codimension 3 linear section of $\mathbb{S}_5$. Then there exists a 2-dimensional subspace K' of $\check{\mathbb{P}}^{15}$ such that $A = \mathbb{S}_5 \cap L_{K'}$, where $L_{K'}$ is the codimension 3 linear subspace of $\mathbb{P}^{15}$ corresponding to K'. Then a codimension 2 smooth linear section X of $\mathbb{S}_5$ containing A corresponds to X_K, where K is a 1-dimensional subspace of $\check{\mathbb{P}}^{15}$ such that $K \subset K'$. We call X_K an *oversetion* of $X_{K'}$.

Since A is locally rigid, thus we may assume that K' is general. In particular, the oversections of A form a 1-dimensional family B_0 in $\mathbb{G}$. Moreover, as K' is general,

$\mathbb{G}$ has Picard number one and R is a divisor in $\mathbb{G}$, we may assume that $B_0 \cap R_0$ is not empty. In particular, there exists a special smooth codimension 2 linear section X of $\mathbb{S}_5$ containing A.

4.B. **VMRT of hyperplanes.** By Theorem 4.4, to prove Theorem 1.3, it remains to consider the case where A is isomorphic to one of the manifolds in the cases (2) and (3) of Theorem 1.2. In particular, A is isomorphic to either a hyperplane section of a raiotional homogeneous space of Picard number one or a hyperplane section of the odd symplectic Grassmannian $\mathrm{Gr}_\omega(2, 2n+1)$ $(n \geq 2)$ and we have a nice description of the VMRT of A.

4.7. Proposition. *Let S be a rational homogeneous space of Picard number one or an odd symplectic Grassmannian $\mathrm{Gr}_\omega(2, 2n+1)$ $(n \geq 2)$. Let A be a locally rigid smooth hyperplane section of S and let $\mathcal{K}$ be a family of minimal rational curves on A. Let $A \subset \mathbb{P}^N$ be the embedding of A given by $|\mathscr{O}_S(A)|_A|$. Then the Hilbert scheme of lines $\mathcal{L}_{o,A}$ of $A \subset \mathbb{P}^N$ coincides with the VMRT $\mathcal{C}_o$ of A at a general point $o \in A$. In particular, the VMRT $\mathcal{C}_o \subset \mathbb{P}(T^*_{o,A})$ is isotrivial over a Zariski open subset of A and it is projectively equivalent to a general hyperplane section of the VMRT $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ at a general point $s \in S$.*

Proof. Let $S \subset \mathbb{P}^{N+1}$ be the embedding of S given by $|\mathscr{O}_S(A)|$. Then $A \subset \mathbb{P}^N$ corresponds to a hyperplane section of $S \subset \mathbb{P}^{N+1}$. Let r is the index of A. By Proposition 4.1, we have $r \geq 3$. It follows that the index of S is at least 4 and $\dim(\mathcal{C}_s) \geq 2$ for a general point $s \in S$. Since S is ruled by lines and $r \geq 3$, A is thus ruled by lines. Since A is locally rigid, we can find a general hyperplane section A_s passing through s such that A_s is isomorphic to A and s is a general point of A_s because of the genericity of s. Moreover, since the Hilbert scheme of lines $\mathcal{L}_{s,S}$ of S coincides with $\mathcal{C}_s$, it follows that the Hilbert scheme of lines $\mathcal{L}_{s,A_s}$ is a general hyperplane section of $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ (see Proposition 3.2). In particular, $\mathcal{L}_{s,A_s}$ is non-degenerate and irreducible because $\dim(\mathcal{C}_s) \geq 2$. As a consequence, $\mathcal{L}_{s,A_s}$ coincides with the VMRT $\mathcal{C}_s$ of A_s at s.

To see the isotriviality of $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,A})$ over a Zariski open subset of A, note that A is quasi-homogeneous if S is not isomorphic to $\mathrm{Gr}(3,8)$ nor an odd symplectic Grassmannian $\mathrm{Gr}_\omega(2, 2n+1)$ $(n \geq 4)$ (see [**?**, Theorem 1.2] and [PVdV99]) and we are done in these cases. In the remaining cases, recall that the VMRT of the Grassmannian $\mathrm{Gr}(3,8)$ is the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^4 \subset \mathbb{P}^{14}$ and the VMRT of the odd symplectic Grassmannian $\mathrm{Gr}_\omega(2, 2n+1)$ at a general point is the rational scroll $S(2, 1^{2n-3}) \subset \mathbb{P}^{4(n-1)}$ over $\mathbb{P}^1$. Then we conclude by the fact that all general hyperplane sections of the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$ (resp. the rational normal scroll $S(2, 1^{2n-3}) \subset \mathbb{P}^{4(n-1)}$ for $n \geq 2$) are projective equivalent. □

4.8. Remarks.

(1) In the case where A is quasi-homogeneous, one can also directly check that all general hyperplane sections of $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ are projective equivalent.

(2) We have seen above that, for a general point $o \in A$, the VMRT $\mathcal{C}_o \subset \mathbb{P}(T^*_{o,A})$ is projectively equivalent to a general hyperplane section of the VMRT $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ of S at a general point s. Note that $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ is an embedding of $\mathcal{C}_s$ given by a complete linear system, thus the VMRT $\mathcal{C}_o \subset \mathbb{P}(T^*_{o,A})$ is also an embedding of $\mathcal{C}_o$ given by a complete linear system.

(3) The general hyperplane section of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ is projective equivalent to the rational normal scroll $S(2, 1^{n-1}) \subset \mathbb{P}^{2n}$. The general hyperplane section of the rational normal scroll $S(2,1) \subset \mathbb{P}^4$ is projective equivalent to the twisted cubic and the general hyperplane section of the rational normal scroll $S(2, 1^n) \subset \mathbb{P}^{2(n+1)}$ $(n \geq 2)$ is projectively equivalent to the rational normal scroll $S(2^2, 1^{n-2}) \subset \mathbb{P}^{2n+1}$.

As an easy corollary of Proposition 4.3, we have the following description of the VMRTs of X.

4.9. Proposition. *Let X be a projective manifold containing a smooth locally rigid ample divisor A, which is isomorphic to a hyperplane section of a rational homogeneous space of Picard number one or an odd symplectic Grassmannian $\mathrm{Gr}_\omega(2, 2k+1)$ $(k \geq 2)$. Let $\mathcal{K}$ be a family of minimal rational curves on X. Denote by $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ the VMRT of X at a general point x. Then a general hyperplane section of $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ is projectively equivalent to the VMRT $\mathcal{C}_o \subset \mathbb{P}(T^*_{o,A})$ at a general point $o \in A$.*

Proof. According to Proposition 4.3, X is a prime Fano manifold with $\mathscr{O}_X(A)$ the ample generator of $\mathrm{Pic}(X)$. Let $X \subset \mathbb{P}^N$ be the embedding given by $|\mathscr{O}_X(A)|$. For a general point $x \in X$, by the locally rigidity of A, we may assume that there exists a general hyperplane section A_x of X passing through x such that A_x is isomorphic to A and the VMRT $\mathcal{C}_{x,A_x} \subset \mathbb{P}(T^*_{x,A_x})$ is projectively equivalent to $\mathcal{C}_o \subset \mathbb{P}(T^*_{o,A})$. Let $\mathcal{L}_x$ be the Hilbert scheme of lines of $X \subset \mathbb{P}^N$ at x. By adjunction formula and Proposition 4.1, we have $r \geq (n+1)/2$, where r is the index of X and $n = \dim(X)$. Then Proposition 3.1 says that the $\mathcal{L}_x$ is smooth. Note that the hyperplane $A_x \subset \mathbb{P}^{N-1}$ is projectively equivalent to the embedding of A given by $|\mathscr{O}_A(1)|$ and therefore the VMRT $\mathcal{C}_{x,A_x} \subset \mathbb{P}(T^*_{x,A_x})$ coincides with the Hilbert scheme of lines of $A_x \subset \mathbb{P}^{N-1}$ at x. Since A_x is a general hyperplane passing through x, by Proposition 3.2, the VMRT $\mathcal{C}_{x,A_x} \subset \mathbb{P}(T^*_{x,A_x})$ is a general hyperplane section of $\mathcal{L}_x \subset \mathbb{P}(T^*_{x,X})$. As a consequence, $\mathcal{L}_x \subset \mathbb{P}(T^*_{x,X})$ is irreducible and non-degenerate. In particular, $\mathcal{L}_x \subset \mathbb{P}(T^*_{x,X})$ coincides with the VMRT $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ and this completes the proof. □

4.10. Remark. Recall that the VMRT $\mathcal{C}_o \subset \mathbb{P}(T^*_{o,A})$ is an embedding of $\mathcal{C}_o$ given by a complete linear system. Then one can see that the VMRT $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ is an embedding of $\mathcal{C}_x$ given by the complete linear system $|\mathscr{O}_{\mathcal{C}_x}(\mathcal{C}_o)|$, where $\mathcal{C}_o$ is regarded as a hyperplane section of $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$.

4.C. **End of the proof.** The following result can be viewed as an analogue of Theorem 1.3 in the case where S is isomorphic to either a product of projective spaces or a special rational scroll over $\mathbb{P}^1$. The argument may be applied to a larger class of scroll over projective spaces, but we will prove only the cases used in this paper.

4.11. Lemma. *Let X be an $(n+1)$-dimensional $(n \geq 2)$ projective manifold containing a smooth ample divisor A such that A is isomorphic to a general smooth member in the tautological class $|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|$ of a projective bundle $\mathbb{P}(\mathscr{E})$, where the vector bundle $\mathscr{E}$ is one of the following*

$$\mathscr{O}_{\mathbb{P}^m}(1)^{\oplus(n-m+2)} (n \geq 2m-1 > 0), \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus n} (n \geq 2).$$

If the restriction $\mathscr{O}_X(A)|_A$ is isomorphic to the restriction $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|_A$, then the polarized pair $(X, \mathscr{O}_X(A))$ is isomorphic to the pair $(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$.

Proof. Since A is isomorphic to a general smooth member of $|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|$, A is a projective bundle over the base $\mathbb{P}^m$. Therefore, there exists a non-splitting sequence of vector bundles

$$0 \to \mathscr{O}_{\mathbb{P}^m} \to \mathscr{E} \xrightarrow{u} \mathscr{Q} \to 0 \tag{4.1}$$

such that $A = \mathbb{P}(\mathscr{Q})$, the inclusion $A \to \mathbb{P}(\mathscr{E})$ is induced by the quotient map u and $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|_A \simeq \mathscr{O}_{\mathbb{P}(\mathscr{Q})}(1)$. Moreover, it is easy to see that A is not isomorphic to the trivial product $\mathbb{P}^m \times \mathbb{P}^{n-m}$. Thus, according to [Liu19, Theorem 1.3], there exists an ample vector bundle $\widehat{\mathscr{E}}$ of rank $n-m+1$ over $\mathbb{P}^m$ such that $X = \mathbb{P}(\widehat{\mathscr{E}})$ and $\mathscr{O}_X(A) \simeq \mathscr{O}_{\mathbb{P}(\widehat{\mathscr{E}})}(1)$, where the natural projection $p\colon A \to \mathbb{P}^m$ is equal to the restriction to A of the bundle projection $\pi\colon \mathbb{P}(\widehat{\mathscr{E}}) \to \mathbb{P}^m$. Then the push-forward of the following exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(A) \to \mathscr{O}_X(A)|_A \to 0$$

by π gives an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^m} \to \widehat{\mathscr{E}} \to \mathscr{Q} \to 0. \tag{4.2}$$

If $\mathscr{E}$ is the vector bundle $\mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus n}$, by (4.1), $\widehat{\mathscr{E}}$ is an ample vector bundle of rank $n+1$ such that $\det(\widehat{\mathscr{E}}) \simeq \mathscr{O}_{\mathbb{P}^1}(n+2)$ and, by Grothendieck's theorem, $\widehat{\mathscr{E}}$ is splitting. As a consequence, it is easy to see that $\widehat{\mathscr{E}}$ is isomorphic to the vector bundle $\mathscr{E}$.

If $\mathscr{E}$ is the vector bundle $\mathscr{O}_{\mathbb{P}^m}(1)^{\oplus(n-m+1)}$, taking the dual sequence of (4.1) yields

$$0 \to \mathscr{Q}^* \to \mathscr{E}^* \to \mathscr{O}_{\mathbb{P}^m} \to 0.$$

This yields a long exact sequence

$$0 \to H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}) \to H^1(\mathbb{P}^m, \mathscr{Q}^*) \to H^1(\mathbb{P}^m, \mathscr{E}^*).$$

By Kodaira's vanishing theorem, we have $h^1(\mathscr{O}_{\mathbb{P}^m}, \mathscr{E}^*) = 0$. Therefore, it follows that

$$H^1(\mathbb{P}^m, \mathcal{Q}^*) \simeq H^0(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}) = \mathbb{C}.$$

Thus, we have

$$\mathrm{Ext}^1(\mathscr{Q}, \mathscr{O}_{\mathbb{P}^m}) = \mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^m}, \mathscr{Q}^*) = \mathrm{H}^1(\mathbb{P}^m, \mathscr{Q}^*) = \mathbb{C}.$$

So the non-trivial extension of $\mathscr{Q}$ by $\mathscr{O}_{\mathbb{P}^m}$ is unique. On the other hand, since $\widehat{\mathscr{E}}$ is an ample vector bundle, the exact sequence (4.2) does not split. Hence, the vector bundle $\widehat{\mathscr{E}}$ is isomorphic to $\mathscr{E}$.

In summary, we have always $\widehat{\mathscr{E}} \simeq \mathscr{E}$ in these two cases and consequently the polarized pair $(X, \mathscr{O}_X(A))$ is isomorphic to the pair $(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$. □

Now we are in the position to prove Theorem 1.3. Note that if G/P is isomorphic to either $Gr(2, 2n)$ or E_6/P_1, then A is a rational homogeneous space and we can conclude by [Wat08, Theorem 1.1]. But we will give a uniform proof by using Theorem 3.3.

Proof of Theorem 1.3. Firstly we remark that a general codimension 2 linear section of $\mathrm{Gr}(2, 2\mathrm{n}+1)$ is isomorphic to a general hyperplane section of $\mathrm{Gr}_\omega(2, 2n+1)$. Thus, by Theorem 4.4, it remains to consider the case where A is isomorphic to a general hyperplane section of one of the following: $\mathrm{Gr}_\omega(2, 2n+1)$ $(n \geq 3)$, $Gr(2, n)$ $(n \geq 6)$, $Gr(3, n)$ $(6 \leq n \leq 8)$, $\mathbb{S}_n$ $(6 \leq n \leq 7)$, E_6/P_1, E_7/P_1. Fix a family $\mathcal{K}$ of minimal rational curves on X. Thanks to Theorem 3.3, it suffices to show that the VMRT $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ at a general point $x \in X$ is projectively equivalent to the VMRT $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$ at a general point $s \in S$, or equivalently it is enough to show that the pair $(\mathcal{C}_x, \mathscr{O}_{\mathcal{C}_x}(1))$ is equivalent to the pair $(\mathcal{C}_s, \mathscr{O}_{\mathcal{C}_s}(1))$.

Recall that the VMRT of E_7/P_7 is E_6/P_1 with its minimal embedding, the VMRT of E_6/P_1 is $\mathbb{S}_5$ with the minimal embedding, and the VMRT of $\mathbb{S}_n$ is $\mathrm{Gr}(2, \mathrm{n})$ with the minimal embedding. Moreover, by Proposition 4.9, there exists an irreducible smooth member in $|\mathscr{O}_{\mathcal{C}_x}(1)|$, which is isomorphic to a general member of $|\mathscr{O}_{\mathcal{C}_o}(1)|$. As a consequence, to prove the Theorem, by induction on VMRT, it suffices to prove it for $\mathrm{Gr}_\omega(2, 2n+1)$ $(n \geq 3)$, $\mathrm{Gr}(2, \mathrm{n})$ $(n \geq 6)$ and $\mathrm{Gr}(3, \mathrm{n})$ $(6 \leq n \leq 8)$.

On the other hand, note that the VMRT of $\mathrm{Gr}_\omega(2, 2n+1)$ and $\mathrm{Gr}(\mathrm{k}, \mathrm{n})$ at a general point are projectively equivalent to the rational normal scroll $S(2, 1^{2n-3}) \subset \mathbb{P}^{4(n-1)}$ and the Segre embedding $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k} \subset \mathbb{P}^{nk-k^2+k-1}$, respectively. Then, according to Proposition 4.9 and Lemma 4.11, we see that the VMRT $\mathcal{C}_x \subset \mathbb{P}(T^*_{x,X})$ is projectively equivalent to the VMRT $\mathcal{C}_s \subset \mathbb{P}(T^*_{s,S})$. It follows from Theorem 3.3 that X is isomorphic to S.

Moreover, if A is not isomorphic to a quadric hypersurface and X' is another projective manifold containing an ample divisor A' isomorphic to A, then X and X' are both isomorphic to a codimension $(k-1)$ locally rigid linear section of S. In particular, X' is isomorphic to X and one can see from Proposition 4.3 that $\mathscr{O}_X(A)$ and $\mathscr{O}_{X'}(A')$ are the ample generators of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X')$, respectively. □

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