CORRIGENDUM TO "FANO MANIFOLDS CONTAINING A NEGATIVE DIVISOR ISOMORPHIC TO A RATIONAL HOMOGENEOUS SPACE OF PICARD NUMBER ONE" [INTERNAT. J. MATH., 2020, 31, 2050066, 14]

JIE LIU

ABSTRACT. In this note, we make a correction to Theorem 1.2 of the aforementioned paper [Liu20].

The purpose of this note is to make a correction to [Liu20]. In [Liu20, Theorem 1.2], we give a classification of pairs (X, A) such that X is a Fano manifold of dimension $n \geq 3$ and A is a smooth ample divisor which is isomorphic to some rational homogeneous space of Picard number 1 and the conormal bundle $\mathscr{N}^*_{A/X}$ is ample. However, it turns out that there exists one case missed in the statement of the theorem and [Liu20, Theorem 1.2] should be read as follows.

0.1. Theorem. *Let X be a Fano manifold of dimension $n \geq 3$ containing a divisor A isomorphic to a rational homogeneous space with Picard number one. Denote by $\mathscr{O}_A(1)$ the ample generator of $\mathrm{Pic}(A)$ and by r the index of A. Assume that $\mathscr{N}_{A/X}$ is isomorphic to $\mathscr{O}_A(-d)$ for some integer $d > 0$. Then $0 < d < r$ and we are in one of the following cases.*

- *(1) $\rho(X) = 2$ and the pair (X, A) is isomorphic to one of the following:*
  - *(1.1) X is isomorphic to $\mathbb{P}(\mathscr{O}_A \oplus \mathscr{O}_A(-d))$ and A is a section with normal bundle $\mathscr{N}_{A/X} \simeq \mathscr{O}_A(-d)$;*
  - *(1.2) X is obtained by blowing up one of the pairs (X', A') listed* [Wat08, Theorem 1] *along a smooth center $C \in |\mathscr{O}_{A'}(d+s)|$, where $\mathscr{O}_{A'}(1)$ is the ample generator of $\mathrm{Pic}(A')$, $\mathscr{N}_{A'/X'} \simeq \mathscr{O}_{A'}(s)$ and A is the strict transform of A'.*
  - *(1.3) X is a smooth element in $|\mathscr{O}_{\mathbb{P}^{n-1}}(\mathscr{E})(2) \otimes \pi^*\mathscr{O}_{\mathbb{P}^{n-1}}(2)|$ and A is isomorphic to a quadric hypersurface such that $X \cap F = A$, where $\mathscr{E}$ is the vector bundle $\mathscr{O}_{\mathbb{P}^{n-1}}(d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$, the map $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^{n-1}$ is the natural projection and the variety $F \subset \mathbb{P}(\mathscr{E})$ is the subbundle corresponding to the quotient $\mathscr{E} \to \mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.*
- *(2) $\rho(X) = 3$ and X is obtained by blowing up a Fano manifold Y along a smooth center $C \in |\mathscr{O}_{A_Y}(d+s)|$ such that $-d < s < r$, where Y is isomorphic to $\mathbb{P}(\mathscr{O}_A \oplus \mathscr{O}_A(s))$, A_Y is a section with normal bundle $\mathscr{N}_{A_Y/Y} \simeq \mathscr{O}_A(s)$, $\mathscr{O}_{A_Y}(1)$ is the ample generator of $\mathrm{Pic}(A_Y)$ and A is the strict transform of A_Y.*

The mistake appears in the proof of [Liu20, Lemma 3.2] and the statement of [Liu20, Lemma 3.2] is false in general. Indeed, in the proof of [Liu20, Lemma 3.2],

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the value of x should be

$$x = \frac{ye}{2+\alpha e} = \frac{e}{2d+(r-d)e},$$

while in the published paper "e" in the denominator disappeared. In particular, the last equation in the same page should be as $2 = 2$ which is trivial. We correct [Liu20, Lemma 3.2] in Lemma 0.2 by proving a weaker statement; that is, the number $2d/e$ is an integer. In particular, for A being a rational homogeneous space of Picard number 1, Lemma 0.2 can be applied to show that A is actually a section of the conic bundle $f : X \to \mathbb{P}^{n-1}$ unless it is isomorphic to a quadric hypersurface or the 10-dimensional spinor variety $\mathbb{S}_5$. Then by a detailed analysis of the conic bundle structure f, we exclude the spinor variety $\mathbb{S}_5$ case by an ad-hoc argument.

Here is the organisation of this short note. In Section 0.A we give an explicit construction of examples for the new case (1.3) of Theorem 0.1. In Section 0.B we prove a weaker statement of [Liu20, Lemma 3.2] to show that $2d/e$ is an integer and then applying it to show that in [Liu20, Lemma 3.2] if A is assumed to be a rational homogeneous space of Picard number 1, then A is a section of f unless A is isomorphic to a quadric hypersurface. Finally we finish the proof of Theorem 0.1 by pointing out the parts affected by [Liu20, Lemma 3.2].

0.A. **Examples.** In this subsection, we construct some examples for case (1.3) of Theorem 0.1. We start from the following example (see [Liu20, Proposition 3.4 (2)]). Let $\mathscr{F} \to \mathbb{P}^n$ be the vector bundle $\mathscr{O}_{\mathbb{P}^n} \oplus \mathscr{O}_{\mathbb{P}^n}(-1)$ with $n \geq 3$. Then $F = \mathbb{P}(\mathscr{F})$ is isomorphic to the blowing-up of $\mathbb{P}^{n+1}$ at a point. Denote by $\mu\colon F \to \mathbb{P}^n$ the blowing-up and let D be the exceptional divisor. Denote by ζ_F the tautological divisor of $\mathbb{P}(\mathscr{F})$ and by $\pi_F : F \to \mathbb{P}^n$ the natural projection. Let H_F be a Weil divisor associated to the pull-back $\pi_F^*\mathscr{O}_{\mathbb{P}^n}(1)$. Then we have

$$\mu^*\mathscr{O}_{\mathbb{P}^n}(1) \cong \mathscr{O}_{\mathbb{P}(\mathscr{F})}(\zeta_F + H_F) \quad \text{and} \quad D = \zeta_F.$$

Let $A \subset F$ be a general smooth member in $|2\zeta_F + 2H_F|$ such that A is disjoint from D. Then A is isomorphic to an n-dimensional quadric hypersurface. Consider the the vector bundle $\mathscr{E} \to \mathbb{P}^n$ which is isomorphic to $\mathscr{O}_{\mathbb{P}^n}(d) \oplus \mathscr{F}$ with $1 \leq d \leq n-1$. Then $F \subset \mathbb{P}(E)$ is a smooth prime divisor. Denote by ζ the tautological divisor of $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^n$ and by H a Weil divisor associated to the pull-back $\pi^*\mathscr{O}_{\mathbb{P}^n}(1)$. Then we have

$$F \in |\zeta - dH|.$$

Recall that the restriction $\zeta|_F$ is isomorphic to ζ_F and $H|_F = H_F$. Consider the following short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\zeta + (d+2)H) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\zeta + 2H) \to \mathscr{O}_F(2\zeta_F + 2H_F) \to 0,$$

As $K_{\mathbb{P}(\mathscr{E})} = -3\zeta + (d-2-n)H$, we have

$$\zeta + (d+2)H = K_{\mathbb{P}(\mathscr{E})} + 4\zeta + (n+4)H.$$

As $d \geq 1$ and $n \geq 3$, $4\zeta + (n+4)H$ is ample. By Kodaira's vanishing theorem, we have $H^1(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(\zeta + (d+2)H) = 0$. In particular, the induced morphism

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\zeta + 2H)) \to H^0(F, \mathscr{O}_F(2\zeta_F + 2H_F))$$

is surjective and there exists a divisor $X \in |2\zeta + 2H|$ such that $X \cap F = A$. Moreover, as A is general and $2\zeta + 2H$ is globally generated, we may assume that X is again smooth. Note that we have

$$\mathscr{O}_X(A) = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(F)|_X \cong \mathscr{O}_X(\zeta - dH).$$

On the other hand, as $\zeta|_A = \zeta_F|_A \sim D|_A = 0$, we get

$$\mathscr{O}_X(A)|_A \cong \mathscr{O}_A(-d).$$

Now we claim that X is a Fano manifold. By adjunction formula, we have

$$K_X = (K_{\mathbb{P}(\mathscr{E})} + 2\zeta + 2H)|_X = \mathscr{O}_X(-\zeta + (d-n)H).$$

If $d \leq n-1$, then $\zeta + (n-d)H$ is a semi-ample big and nef divisor with non-ample locus contained in D. By our construction, the variety X is disjoint from D, thus the restriction $(\zeta + (n-d)H)|_X$ is ample and hence $-K_X$ is ample.

0.B. **Correction of [Liu20, Lemma 3.2].** As pointed out in the beginning, [Liu20, Lemma 3.2] is not correct in general. We replace it by the following weaker statement.

0.2. Lemma. *Let X be a Fano manifold of dimension $n \geq 3$ and with $\rho(X) = 2$, and let A be a smooth Fano hypersurface of X such that $\mathrm{Pic}(A) \simeq \mathbb{Z}\mathscr{O}_A(1)$ for some ample line bundle $\mathscr{O}_A(1)$ and $\mathscr{N}_{A/X} \simeq \mathscr{O}_A(-d)$ for some $d > 0$. Assume furthermore that there exists a curve of degree 1 on A; i.e. an irreducible curve $C \subset A$ such that $c_1(\mathscr{O}_A(1)) \cdot C = 1$. If X admits an extremal contraction $f\colon X \to \mathbb{P}^{n-1}$, which is a conic bundle, such that f is finite over A. Then $f^*\mathscr{O}_{\mathbb{P}^{n-1}}(1) \cong \mathscr{O}_A(1)$ and $2d/e$ is an integer, where $e = \deg_A(\mathscr{O}_A(1))$.*

Proof. Denote by r the index of A, i.e., $\mathscr{O}_A(-K_A) \simeq \mathscr{O}_A(r)$. As X is Fano, the line bundle $\mathscr{O}_A(-K_X) \simeq \mathscr{O}_A(r-d)$ is ample. We get $r > d$. Since A is not nef and X is Fano, there exists an extremal ray R of $\overline{\mathrm{NE}}(X)$ such that $A \cdot R < 0$. Let $\pi\colon X \to Y$ be the associated contraction. Then $\mathrm{Exc}(\pi) \subset A$ as $A \cdot R < 0$. On the other hand, every curve contained in A has class in R since $\rho(A) = 1$ and $\mathscr{O}_X(A)|_A \simeq \mathscr{N}_{A/X}$ is negative. This implies that $A = \mathrm{Exc}(\pi)$ and that $\pi(A)$ is a point. By adjunction, we have

$$K_X \sim_{\mathbb{Q}} \pi^* K_Y + \frac{r-d}{d} A.$$

Let H be a Weil divisor associated to $f^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. Since $\rho(X) = 2$ and the contraction map f is supposed to be elementary, there exist $x, y \in \mathbb{Q}$ such that

$$H \equiv x\pi^*(-K_Y) - yA.$$

Denote by e the degree $\mathscr{O}_A(1)^{n-1}$. Set $\alpha = (r-d)/d$ and $m{:=}(-K_Y)^n$. Then we get

$$0 = H^n = x^n m - y^n d^{n-1} e \tag{0.1}$$

and

$$2 = (-K_X) \cdot H^{n-1} = x^{n-1} m - \alpha y^{n-1} d^{n-1} e. \tag{0.2}$$

Set $l{:=}2d^2/(yd)^{n-1}$. Now we follow the argument of [Tsu06, Lemma 1] to show that $yd = 1$. By (0.1), we have

$$\left(\frac{y}{x}\right)^n = \frac{m}{d^{n-1}e}.$$

Combining (0.1) and (0.2) yields

$$x = \frac{y^n d^{n-1} e}{2 + \alpha y^{n-1} d^{n-1} e}.$$

This implies

$$\frac{y}{x} = y \cdot \frac{2 + \alpha y^{n-1} d^{n-1} e}{y^n d^{n-1} e} = \frac{2 + \alpha y^{n-1} d^{n-1} e}{y^{n-1} d^{n-1} e}.$$

It follows

$$\left(\frac{2d^2}{y^{n-1} d^{n-1}} + \alpha d^2 e \right)^n = \frac{m}{d^{n-1} e} \cdot (d^2 e)^n = m d^n e^{n-1} \cdot d.$$

As md^n is an integer, it follows that

$$\frac{2d^2}{y^{n-1} d^{n-1}} + \alpha d^2 e = \frac{2d^2}{y^{n-1} d^{n-1}} + (r-d) de$$

is an integer. In particular, l is an integer. As $d < n$, we obtain

$$2(n-1)^2 \geq 2d^2 = (yd)^{n-1} \cdot l.$$

As $H \cdot C = -yA \cdot C = -yc_1(\mathscr{O}_A(A)) \cdot C = yd$ is an integer and $n \geq 3$, we must have $yd \leq 2$. Moreover, if $yd = 2$, we have $2d^2 = 2^{n-1} \cdot l$, hence $d^2 = 2^{n-2} \cdot l$. On the other hand, as $(l + (r-d)de)^n = md^n e^{n-1} d \in d\mathbb{N}$, we have $l^n \in d\mathbb{N}$. In particular, as $d \leq n-1$, we obtain $(n, d, l) \in \{(3, 2, 2), (5, 4, 2)\}$. If $(n, d, l) \in \{(3, 2, 2), (5, 4, 2)\}$, then we must have $r > d = n - 1 = \dim(A)$. By Kobayashi-Ochiai's theorem, then A is isomorphic to $\mathbb{P}^{n-1}$, which is impossible by [Tsu06, Lemma 1]. Thus, we have $yd = 1$ and as a consequence, we have

$$A \cdot H^{n-1} = A \cdot (x\pi^*(-K_Y) - yA)^{n-1} = (yd)^{n-1} e = e.$$

In particular, we get $H|_A \cong \mathscr{O}_A(1)$. As a consequence, we obtain

$$x = \frac{ye}{2 + \alpha e} = \frac{e}{2d + (r-d)e} \quad \text{and} \quad \frac{y}{x} = \frac{2 + \alpha e}{e} = \frac{2d + (r-d)e}{ed}.$$

It yields

$$K_X = -\frac{1}{x} H - \frac{y}{x} A + \frac{r-d}{d} A = -\frac{2d + (r-d)e}{e} H - \frac{2}{e} A.$$

This implies

$$\begin{aligned} K_X^2 \cdot H^{n-2} &= 2 \cdot \left[\frac{2d + (r-d)e}{e} \right] \cdot \left(\frac{2}{e} \right) A \cdot H^{n-1} + \left(\frac{2}{e} \right)^2 A^2 \cdot H^{n-2} \\ &= 4 \left[\frac{2d + (r-d)e}{e} \right] - \frac{4d}{e} \\ &= \frac{4d}{e} + 4(r-d). \end{aligned}$$

As $K_X^2 \cdot H^{n-2}$ is an integer, it follows that $4d/e$ is an integer. On the other hand, set $\beta = [2d + (r-d)e]/e$, we also have

$$\begin{aligned}(-K_X)^3 \cdot H^{n-3} &= 3\beta^2 \cdot \frac{2}{e} A H^{n-1} + 3\beta \cdot \left(\frac{2}{e}\right)^2 A^2 \cdot H^{n-2} + \left(\frac{2}{e}\right)^3 A^3 \cdot H^{n-3} \\ &= 6\beta^2 - \frac{12\beta d}{e} + \frac{8d^2}{e^2} \\ &= 6\left[\frac{2d+(r-d)e}{e}\right]^2 - \frac{12d}{e} \cdot \left[\frac{2d+(r-d)e}{e}\right] + \frac{8d^2}{e^2} \\ &= \frac{24d^2}{e^2} + \frac{24d(r-d)}{e} + 6(r-d)^2 - \frac{24d^2}{e^2} - \frac{12d(r-d)}{e} + \frac{8d^2}{e^2} \\ &= \frac{12d(r-d)}{e} + 6(r-d)^2 + \frac{8d^2}{e^2}.\end{aligned}$$

As $(-K_X)^3 \cdot H^{n-2}$ and $4d/e$ are integers, it follows that $8d^2/e^2$ is an integer. This implies that $2d/e$ is an integer. In particular, we have $e \leq 2d \leq 2r - 2$. □

The remaining part of this section is devoted to prove the following result, which will be used to finish the proof of Theorem 0.1.

0.3. Proposition. *In Lemma 0.2, if we assume in addition that A is isomorphic to a rational homogeneous space of Picard number 1, then A is a section of f unless A is isomorphic to a quadric hypersurface.*

The proof of Proposition 0.3 above will be divided into two different parts. In the first part, we show that a rational homogeneous space A of Picard number 1 satisfies $e \leq 2r-2$ if and only if it is isomorphic to one of the following: a projective space, a quadric hypersurface, the Grassmann variety $\mathrm{Gr}(2,5)$ and the 10-dimensional spinor variety $\mathbb{S}_5$. The projective space cases are proved in [Tsu06] and the Grassmann variety $\mathrm{Gr}(2,5)$ can be easily excluded by the fact that $2d/e$ is an integer for some $d \leq r-1$. In the second part, we exclude the spinor variety $\mathbb{S}_5$ case by studying the conic bundle structure f carefully.

0.B.1. *Rational homogeneous space of small degrees.* Now we proceed to classify rational homogeneous spaces of Picard number 1 satisfying $e \leq 2r-2$.

0.4. Proposition. *Let A be an n-dimensional rational homogeneous space of Picard number 1 with degree e and index r. Then $e \leq 2r-2$ if and only if A is isomorphic to one of the following varieties:*

(1) a projective space $\mathbb{P}^n$ with $e = 1$ and $r = n+1$;
(2) a quadric hypersurface $\mathbb{Q}^n$ $(n \geq 3)$ with $e = 2$ and $r = n$;
(3) the Grassmann variety $\mathrm{Gr}(2,5)$ with $e = 5$ and $r = 5$;
(4) the 10-dimensional spinor variety $\mathbb{S}_5$ with $e = 12$ and $r = 8$.

0.5. Theorem. [Ion08] *Let $Z \subsetneq \mathbb{P}^N$ be an n-dimensional irreducible, smooth, non-degenerate and linearly normal projective variety of degree e. Assume that Z is a Fano manifold of Picard number 1 such that $2N \geq 3n$ and $n \geq 2$. If $e \leq N$, then Z has index at least $n-2$.*

Proof. Denote by $c = N - n$ the codimension of Z. Then we have $n \leq 2c$ by assumption. Firstly we assume that $n \leq c+1$. Then we have

$$e \leq N \leq N + c + 1 - n = 2c + 1.$$

As Z has Picard number 1, it follows from [Ion85, Theorem I] that Z has index at least $n-1$.

Secondly we assume that $c+2 \leq n \leq 2c$. Let Δ be the Δ-genus $e-c-1$ of Z. If $\Delta \leq 1$, it is well known from the classification of Fano manifolds that X is has index $\geq n-1$ (see for instance [Ion08, Theorem A and B]). Thus we may assume that $\Delta \geq 2$. Then it follows from [Ion08, Propoistion 10] that Z has index $n-2$. □

0.6. Lemma. *Let A be an n-dimensional rational homogeneous space of Picard number 1. If $r \geq n-2$, then A is isomorphic to one of the following*

$$\mathbb{P}^n,\ Q^n\ (n \geq 3),\ \mathrm{Gr}(2,5),\ \mathbb{S}_5,\ \mathrm{Gr}(2,6),\ LG(3,6),\ G_2/P_2.$$

In particular, $e \leq 2r-2$ if and only A is isomorphic to one of the varieties listed in Proposition 0.4.

Proof. This is well-known from the classification of Fano manifold of index at least $n-2$, see [IP99, Theorem 3.1.14, Table 12.1 and Theorem 5.2.3]. In particular, the corresponding pairs (e,r) are as follows

$$(1,n+1),\ (2,n),\ (5,5),\ (12,8),\ (14,6),\ (16,4)\ (18,3).$$

This finishes the proof. □

0.7. Lemma. *Let $X = \mathcal{D}_l/P_k$ be a rational homogeneous space of Picard number 1, with index r. Let L be the ample generator of $\mathrm{Pic}(X)$. Then $2r > h^0(X,L)+1$ if and only if X is isomorphic to either $\mathbb{P}^n$ or $\mathbb{Q}^n$ $(n \geq 4)$.*

Proof. We refer the reader to [Kon86, Table 1] for the explicit values of r and $h^0(X,L)$ in terms of l and k. We just remark that in [Kon86, Table 1], the index of X is denoted by k and the node is denote by r. Moreover, we also recall that G_2/P_1 is isomorphic to the 5-dimensional quadric hypersurface $\mathbb{Q}^5$. In particular, if X is of E-F-G type, it can be easily shown that $2r > h+1$ if and only if X is isomorphic to G_2/P_1, where $h = h^0(X,L)$. Now we prove it for X being of classical type. In the following table, we collect the values of r and h for X of classical types. Here we remark that B_l/P_l is isomorphic to D_l/P_{l-1} and it is also isomorphic to D_l/P_l which is called the spinor variety $\mathbb{S}_l$, and C_l/P_1 is isomorphic to A_{2l-1}/P_1 which is the projective space $\mathbb{P}^{2l-1}$.

$\mathcal{D}_l$	node k	r	h
A_l	$1 \leq k \leq l$	$l+1$	$\binom{l+1}{k}$
B_l	$1 \leq k \leq l-1$	$2l-k$	$\binom{2l+1}{k}$
C_l	$2 \leq k \leq l$	$2l+1-k$	$\binom{2l}{k} - \binom{2l}{k-2}$
D_l	$1 \leq k \leq l-2$	$2l-1-k$	$\binom{2l}{k}$
D_l	$l-1$	$2l-2$	2^{l-1}

(1) $\mathcal{D}_l = A_l$. Firstly we note that A_l/P_k is isomorphic to A_l/P_{l-k+1}. Thus we may assume that $2k \leq l+1$. Moreover, A_l/P_1 is isomorphic to the projective space $\mathrm{Gr}(1,l+1) = \mathbb{P}^l$ and A_3/P_2 is isomorphic to $\mathrm{Gr}(2,4)$ which is the 4-dimensional quadric hypersurface. For $2 \leq k \leq \frac{l+1}{2}$, by our assumption, we

have
$$2r = 2(l+1) > h = \binom{l+1}{k} \geq \binom{l+1}{2} = \frac{l(l+1)}{2}.$$
This implies that $l = 3$ and $k = 2$; that is, X is isomorphic to $\mathbb{Q}^4$.

(2) $\mathcal{D}_l = B_l$. Firstly we note that B_l/P_1 is isomorphic to the $(2l-1)$-dimensional quadric hypersurface $\mathbb{Q}^{2l-1}$. If $k \geq 2$, by our assumption, we have
$$4l - 4 \geq 4l - 2k = 2r > h = \binom{2l+1}{k} \geq \binom{2l+1}{2} = l(2l+1) \geq 4l + 2,$$
which is obviously impossible.

(3) $\mathcal{D}_l = C_l$. Firstly we note that C_2/P_2 is isomorphic to the 3-dimensional quadric hypersurface. By our assumption, we have $2r \geq h + 2 \geq \dim(X) + 3$ since L is very ample. Recall that the dimension of X is as follows:
$$\dim(X) = 2k(l-k) + \frac{k(k+1)}{2}.$$
Thus, if $2 \leq k \leq l-1$, then we have
$$\begin{aligned} 2r = 4l + 2 - 2k &\geq 2k(l-k) + \frac{k(k+1)}{2} + 3 \\ &\geq (2k-4)l + 4l - 2k^2 + \frac{k(k+1)}{2} + 3 \\ &\geq (2k-4)(k+1) + 4l - 2k^2 + \frac{k(k+1)}{2} + 3 \\ &\geq 4l - 1 + \frac{k(k-3)}{2} \end{aligned}$$
which is possible only if $k = 2$. Nevertheless, if $k = 2$, then we have $2r = 4l - 2$ and $h + 1 = l(2l-1)$, which is impossible as $l \geq k + 1 = 3$. Thus we may assume that $l = k$. Then we obtain
$$2r = 2l + 2 > h + 1 \geq \dim(X) + 2 = \frac{l(l+1)}{2} + 2,$$
which is impossible unless $l = 2$. On the other hand, note that C_2/P_2 is isomorphic to the 3-dimensional quadric hypersurface, which is again impossible.

(4) $\mathcal{D}_l = D_l$ and $1 \leq k \leq l-2$. Firstly we note that D_l/P_1 is the $(2l-2)$-dimensional quadric hypersurface. For $k \geq 2$, by our assumption, we have
$$4l - 6 \geq 2r = 2(2l-1-k) > h = \binom{2l}{k} \geq \binom{2l}{2} = l(2l-1),$$
which is impossible as $l \geq k + 2 \geq 4$.

(5) $\mathcal{D}_l = D_l$ and $k = l-1$. If $2 \leq l \leq 4$, the variety X is isomorphic to $\mathbb{P}^1$ $(l = 2)$, $\mathbb{P}^3$ $(l = 3)$ and the 6-dimensional quadric hypersurface $\mathbb{Q}^6$ $(l = 4)$. Thus, we may assume that $l \geq 5$. Then by our assumption, we obtain
$$2r = 2(2l-2) > h = 2^{l-1} = 4 \cdot 2^{l-3} \geq 4 \cdot 2(l-3),$$
which is impossible.

This finishes the proof. □

Now we are in the position to prove Proposition 0.4.

Proof of Proposition 0.4. Let L be the ample generator of $\mathrm{Pic}(A)$. Then L is very ample. Denote $h^0(X, L)$ by h.

If $2r > h+1$, by Lemma 0.7, A is isomorphic to either a projective space or a quadric hypersurface.

If $2r \leq h+1$, then we get $e \leq 2r-2 \leq h-1$ and therefore Theorem 0.5 implies that either A has index $\geq n-2$, or $2(h-1) < 3n$. In the former case, we can conclude by lemma 0.6. In the latter case, we note that A is quadric; that is, the embedding $A \subset \mathbb{P}(H^0(A, L))$ is scheme-theoretically cut out by quadric hypersurfaces. Then A is actually a complete intersection in $\mathbb{P}(H^0(A, L))$ (cf. [IR13]). Hence, A is actually a quadric hypersurface. □

0.B.2. *Fano conic bundles.* Let $f : X \to \mathbb{P}^{n-1}$ be an n-dimensional Fano conic bundle with $n \geq 3$, i.e., X is a Fano manifold and f is a conic bundle structure. Denote by $\mathscr{E}$ the locally free sheaf $f_*\mathscr{O}_X(-K_X)$ of rank 3. Let ζ be the tautological divisor of $\mathbb{P}(\mathscr{E})$. Denote by c the integer such that $\det(\mathscr{E}) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(c)$. Let H be a Weil divisor associated to $\pi^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$, where $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^{n-1}$ is the natural projection. Then X can be embedded in $\mathbb{P}(\mathscr{E})$ as a divisor such that

$$X \in |2\zeta + (n-c)H|.$$

Let $A \subset X$ be an irreducible smooth divisor which is a Fano manifold of Picard number 1 such that $H|_A$ is the ample generator of $\mathrm{Pic}(A)$ and $\mathscr{O}_X(A)|_A \cong \mathscr{O}_A(-dH)$ for some $d > 0$. Denote by e the degree of A with respect to $H|_A$ and by $h : A \to \mathbb{P}^{n-1}$ the induced finite morphism. Let r be the index of A.

0.8. Lemma. *Let $\mathscr{E} \to \mathscr{O}_{\mathbb{P}^{n-1}}(a)$ be a non-zero morphism of coherent sheaves. If $a \leq 0$, then there exists an integer $b \leq a$ such that $2b = c - n$ and*

$$\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(c-b-r+d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(r-d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(b).$$

Proof. Let $\mathscr{Q} \subset \mathscr{O}_{\mathbb{P}^{n-1}}(a)$ be the image of $\mathscr{E}$ and denote by $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(b)$ the reflexive hull of $\mathscr{Q}$. Then we have $b \leq a \leq 0$. In particular, the generically surjective morphism $\mathscr{E} \to \mathscr{L}$ defines a rational section $S \subset \mathbb{P}(\mathscr{E})$ such that there exists a Zariski open subset $U \subset \mathbb{P}^{n-1}$ satisfying

(1) $\mathrm{codim}(\mathbb{P}^{n-1} \setminus U) \geq 2$;
(2) $S \cap \pi^{-1}(U) \to U$ is an isomorphism;
(3) $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(\zeta)|_{S\cap\pi^{-1}(U)} \cong \pi^*\mathscr{L}|_{S\cap\pi^{-1}(U)}$.

Take a log resolution $\mu : \widetilde{S} \to S$ such that μ is an isomorphism over $S \cap \pi^{-1}(U)$ and denote by $g : \widetilde{S} \to \mathbb{P}^{n-1}$ the induced birational morphism. Then we have

$$\mu^*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(\zeta) \cong g^*\mathscr{O}_{\mathbb{P}^{n-1}}(b) \otimes \mathscr{O}_{\widetilde{S}}(\Delta),$$

where Δ is a g-exceptional divisor. Since ζ is π-ample, the pull-back $\mu^*\zeta$ is g-nef. Then the negativity lemma implies that $-\Delta$ is effective.

Claim 1. *Let $C \subset S$ be an irreducible projective curve such that $C \cap \pi^{-1}(U) \neq \emptyset$. Then we have $\zeta \cdot C \leq 0$.*

Proof of Claim 1. By assumption, the intersection $\mu(\Delta) \cap \pi^{-1}(U)$ is empty. Let $\widetilde{C} \subset \widetilde{S}$ be the strict transform of C. Then we have

$$\zeta \cdot C = \mu^*\zeta \cdot \widetilde{C} = c_1(g^*\mathscr{O}_{\mathbb{P}^{n-1}}(b)) \cdot \widetilde{C} + \Delta \cdot \widetilde{C} \leq b \leq 0.$$

This finishes the proof of Claim 1.

Note that $\zeta|_X = -K_X$ is ample, thus Claim 1 implies that the image $\pi(X \cap S)$ is contained in $\mathbb{P}^{n-1} \setminus U$. In particular, let $l \subset U$ be a general line and let $\bar{l} \subset S$ be the section corresponding to the quotient $\mathscr{E}|_l \to \mathscr{L}|_l$. Then X is disjoint from $\bar{l}$. In particular, we have

$$X \cdot \bar{l} = (2\zeta + (n-c)H) \cdot \bar{l} = 2b + (n-c) = 0.$$

As a consequence, we have $2b = c - n$.

Claim 2. *The morphism $\mathscr{E} \to \mathscr{L}$ is surjective.*

Proof of Claim 2. Let $x \in X$ be an arbitrary point and let $x \in l$ be a general line passing through x such that $l \cap U \neq \emptyset$. We consider the restriction

$$\sigma_l : \mathscr{E}|_l \to \mathscr{L}|_l.$$

We claim that σ_l is surjective. Otherwise, let $\mathscr{Q}_l$ be the image of σ_l. Then we must have $\mathscr{Q}_l \cong \mathscr{O}_{\mathbb{P}^1}(b')$ for some $b' < b$. Let $\bar{l} \subset \mathbb{P}(\mathscr{E}|_l)$ be the section corresponding to the quotient $\mathscr{E}|_l \to \mathscr{Q}_l$. Then we obtain

$$X \cdot \bar{l} = (2\zeta + (n-c)H) \cdot \bar{l} = 2b' + n - c < 2b + n - c = 0.$$

In particular, $\bar{l}$ is contained in X and $\zeta \cdot \bar{l} = b' < 0$, which is impossible as $\zeta|_X = -K_X$ is ample. This finishes the proof of claim 2.

Claim 3. *The vector bundles $\mathscr{E}$ splits as a direct sum of line bundles as follows*

$$\mathscr{O}_{\mathbb{P}^{n-1}}(c - b - r + d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(r - d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(b).$$

Proof of Claim 3. Firstly note that we have $\zeta|_A \cong \mathscr{O}_A(r-d)$. Thus $h^*\mathscr{E}$ admits a quotient line bundle $h^*\mathscr{E} \to \mathscr{O}_A(r-d)$ with the corresponding section $A' \subset \mathbb{P}(h^*\mathscr{E})$ such that

$$\bar{h}(A') = A,$$

where $\bar{h} : \mathbb{P}(h^*\mathscr{E}) \to \mathbb{P}(\mathscr{E})$ is the induced morphism.

On the other hand, let $S' \subset \mathbb{P}(h^*\mathscr{E})$ be the section corresponding to the induced quotient line bundle $h^*\mathscr{E} \to h^*\mathscr{L}$. Then we have $\bar{h}(S') = S$. By Claim 2, S is a section of $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^{n-1}$ such that $\zeta|_S \cong \mathscr{O}_{\mathbb{P}^{n-1}}(b)$. This yields that X is disjoint from S and hence A is disjoint from S. Thus, A' is also disjoint from S'. Let $\mathscr{F} \subset \mathscr{E}$ be the kernel of $\mathscr{E} \to \mathscr{L}$. Then the induced morphism $h^*\mathscr{F} \to \mathscr{O}_A(r-d)$ is surjective. As a consequence, we obtain the following exact sequence of vector bundles

$$0 \to \mathscr{G} \to h^*\mathscr{F} \to \mathscr{O}_A(r-d) \to 0.$$

As A is a Fano manifold of Picard number 1 and of dimension ≥ 2, we must have $H^1(A, \mathscr{O}_A(i)) = 0$ for any $i \in \mathbb{Z}$ by Kodaira's vanishing theorem. Then we obtain

$$h^*\mathscr{F} \cong \mathscr{G} \oplus \mathscr{O}_A(r-d) \cong \mathscr{O}_A(c-b-r+d) \oplus \mathscr{O}_A(r-d).$$

Then Lemma 0.9 below implies that $\mathscr{F} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(c-b-r+d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(r-d)$ and we are done. □

0.9. Lemma. *Let $f : Y \to X$ be a finite morphism between Fano manifolds of Picard number 1 with dimension at least 2. Let $\mathscr{E}$ be a vector bundle of rank 2 over X. If $f^*\mathscr{E} \cong \mathscr{L}_1 \oplus \mathscr{L}_2$, then there exist line bundles $\mathscr{M}_i$ on X such that $f^*\mathscr{M}_i \cong \mathscr{L}_i$ for $i = 1, 2$ and $\mathscr{E} \cong \mathscr{M}_1 \oplus \mathscr{M}_2$.*

Proof. Firstly we assume that $f^*\mathscr{E}$ is semistable. Then we have $\mathscr{L}_1 \cong \mathscr{L}_2$ as Y is Fano with $\rho(Y) = 1$. In particular, the vector bundle $f^*\mathscr{E}$ is numerically projectively flat (see [LOY20, Definition 4.1]), so is $\mathscr{E}$ itself. As X is simply connected, it follows that $\mathscr{E}$ is a direct sum $\mathscr{M}_1 \oplus \mathscr{M}_2$ such that $\mathscr{M}_1 \cong \mathscr{M}_2$. Then it is clear that we have $f^*\mathscr{M}_1 \cong \mathscr{L}_1$ as $\det(f^*\mathscr{E}) = f^*\det(\mathscr{E})$.

Next we assume that $f^*\mathscr{E}$ is not semistable. Then $\mathscr{E}$ itself is not semistable. Without loss of generality, we may assume that $c_1(\mathscr{L}_1) > c_1(\mathscr{L}_2)$. Let $\mathscr{M}_1 \subset \mathscr{E}$ be the maximal destabilisor. Then we have $c_1(f^*\mathscr{M}_1) > c_1(\mathscr{L}_2)$. In particular, the induced morphism $f^*\mathscr{M}_1 \to f^*\mathscr{E}$ factors through $\mathscr{L}_1 \to f^*\mathscr{E}$; that is, $f^*\mathscr{M}_1 \subset \mathscr{L}_1$. As $\mathscr{M}_1$ is an invertible sheaf and $\mathscr{M}_1$ is saturated in $\mathscr{E}$, it follows that $f^*\mathscr{M}_1 \subset \mathscr{L}_1$ is also saturated and hence $f^*\mathscr{M}_1 \to \mathscr{L}_1$ is an isomorphism. Thus, the line bundle $\mathscr{M}_1$ is a subbundle of $\mathscr{E}$ and therefore $\mathscr{M}_2 := \mathscr{E}/\mathscr{M}_1$ is a line bundle satisfying $f^*\mathscr{M}_2 \cong \mathscr{L}_2$. In particular, as X is a Fano manifold of Picard number 1 with dimension at least 2, it follows $H^1(X, \mathscr{M}) = 0$ for any line bundle $\mathscr{M}$ over X, and hence $\mathscr{E} \cong \mathscr{M}_1 \oplus \mathscr{M}_2$. □

Now we assume that A is the 10-dimensional spinor variety. As $2d/e$ is an integer, $e = 12$ and $d + 1 \leq r = 8$, as computed in the proof of Lemma 0.2, we obtain

$$e = 12, d = 6, r = 8, \mathscr{O}_A(\zeta) \cong \mathscr{O}_A(2) \text{ and } \mathscr{O}_X(A) \cong \mathscr{O}_X(6\zeta - 18H).$$

Moreover, we have the following equations:

$$\begin{cases} K_X^2 \cdot (H|_X)^9 & = \frac{4d}{e} + 4(r-d) = 10 \\ (-K_X)^3 \cdot (H|_X)^8 & = \frac{12d(r-d)}{e} + 6(r-d)^2 + \frac{8d^2}{e^2} = 38 \\ (-K_X)^4 \cdot (H|_X)^7 & = (-K_X)^3 \cdot (3H|_X + \frac{1}{6}A) \cdot (H|_X)^7 = 130. \end{cases}$$

Denote by L a general hyperplane section of $\mathbb{P}^{10}$. We are ready to calculate the Chern classes of $\mathscr{E}$. Recall that we have the following

$$\zeta^3 = \pi^* c_1(\mathscr{E}) \cdot \zeta^2 - \pi^* c_2(\mathscr{E}) \cdot \zeta + \pi^* c_3(\mathscr{E}).$$

Firstly we have

$$\zeta^2 \cdot (2\zeta + (11-c)H) \cdot H^9 = K_X^2 \cdot (H|_X)^9 = 10.$$

This implies that $11 + c = 10$ and hence $c = \zeta^3 \cdot H^9 = -1$.

Secondly we have

$$\zeta^3 \cdot (2\zeta + 12H) \cdot H^8 = (-K_X)^3 \cdot (H|_X)^8 = 38.$$

This implies that $c_2(\mathscr{E}) \cdot L^8 = -24$. One can also calculate that $c_3(\mathscr{E}) \cdot L^7 = -36$, but we do not need it in the following so we leave it for the interested reader.

0.10. Proposition. *In Lemma 0.2, A is not isomorphic to the 10-dimensional spinor variety $\mathbb{S}_5$.*

Proof. As $c_2(\mathscr{E}) \cdot L^8 < 0$, the Bogomolov inequality implies that $\mathscr{E}$ is not semistable. Let $\mathscr{Q}$ the last graded piece of the Harder-Narasimhan filtration of $\mathscr{E}$ and denote by $\mathscr{G}$ the quotient $\mathscr{E}/\mathscr{Q}$. Then the determinant $\det(\mathscr{G})$ is isomorphic to $\mathscr{O}_{\mathbb{P}^{n-1}}(b)$ for some $b \leq -1$.

Firstly we assume that $\mathscr{G}$ has rank 1. Then by Lemma 0.8 above, $b = -6$ and we have

$$\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^{10}}(3) \oplus \mathscr{O}_{\mathbb{P}^{10}}(2) \oplus \mathscr{O}_{\mathbb{P}^{10}}(-6).$$

Let $h^*\mathscr{E} \to \mathscr{O}_A(2)$ be the line bundle quotient corresponding to a section $A' \subset \mathbb{P}(\mathscr{E})$ such that $\bar{h}(A') = A$. Then it is clear that we have the following factorisation

$$h^*\mathscr{E} \to \mathscr{O}_A(2) \oplus \mathscr{O}_A(-6) \to \mathscr{O}_A(2).$$

This implies that A is contained in the prime divisor

$$F = \mathbb{P}(\mathscr{O}_{\mathbb{P}^{10}}(2) \oplus \mathscr{O}_{\mathbb{P}^{10}}(-6)) \subset \mathbb{P}(\mathscr{E}).$$

Note that $F \cap X \to \mathbb{P}^{10}$ is a generically finite morphism of degree 2 since X is a conic bundle and $F \in |\zeta - 3H|$. Nevertheless, this is impossible as A is an irreducible component of $F \cap X$ and $A \to \mathbb{P}^{10}$ is of degree $e = 12$.

Now we assume that $\mathscr{G}$ has rank 2. Let $\mathscr{L}$ be the kernel of $\mathscr{E} \to \mathscr{G}$. Then we have $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^{10}}(a)$ for some $a \geq 0$ by the construction of $\mathscr{G}$.

Claim. $a \leq 2$.

Proof of Claim. Assume to the contrary that $a > 2$. By our assumption, there exists a line bundle quotient $h^*\mathscr{E} \to \mathscr{O}_A(2)$ with the corresponding section $A' \subset \mathbb{P}(h^*\mathscr{E})$ such that $\bar{h}(A') = A$. Moreover, as $a > 2$, it follows that the composition

$$h^*\mathscr{L} \to h^*\mathscr{E} \to \mathscr{O}_A(2)$$

is identically zero. Hence, we have a factorisation

$$h^*\mathscr{E} \to h^*\mathscr{G} \to \mathscr{O}_A(2).$$

Let $F \subset \mathbb{P}(\mathscr{E})$ be the main component of $\mathbb{P}(\mathscr{G}) \subset \mathbb{P}(\mathscr{E})$. Then F is a prime divisor such that $F \in |\zeta - aH|$ and $A \subset F$. As before, the induced morphism $F \cap X \to \mathbb{P}^{10}$ is a generically finite morphism of degree 2, while $A \to \mathbb{P}^{10}$ is of degree 12, which is impossible. This finishes the proof of the claim.

Note that $\mathscr{G}$ is semistable by our assumption. Thus the Bogomolov's inequality says that $c_2(\mathscr{G}) \cdot L^8 \geq 0$ (see [HL10, Theorem 3.4.1]). Nevertheless, by the definition of Chern classes, we have

$$c_2(\mathscr{G}) \cdot L^8 + c_1(\mathscr{G}) \cdot c_1(\mathscr{L}) \cdot L^8 = c_2(\mathscr{E}) \cdot L^8 = -24.$$

This implies

$$c_2(\mathscr{G}) \cdot L^8 = -24 - (-1-a)a = -24 + a + a^2 \leq -18,$$

which is a contradiction. ☐

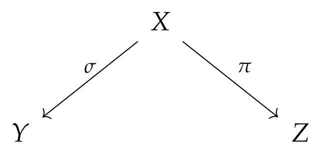
0.11. Remark. One can see that the direct sum $\mathscr{O}_{\mathbb{P}^{10}}(2) \oplus \mathscr{O}_{\mathbb{P}^{10}}(3) \oplus \mathscr{O}_{\mathbb{P}^{10}}(-6)$ has Chern classes $(-1, -24, -36)$ with respect to L.

Now we are in the position to prove Proposition 0.3.

Proof of Proposition 0.3. By Lemma 0.2 and Proposition 0.4, the only possibilities of A are as follows: a projective space, a quadric hypersurface, the Grassmann variety $\mathrm{Gr}(2,5)$ and the 10-dimensional spinor variety $\mathbb{S}_{10}$. If A is a projective space, then it is proved in [Tsu06] that A is a section of f. If A is the Grassmann variety $\mathrm{Gr}(2,5)$, then we have $e = r = 5$. In particular, there does not exists positive integers $d \leq r-1$ such that $2d/e$ is an integer and we can exclude it by Lemma 0.2. The 10-dimensional spinor variety $\mathbb{S}_5$ is excluded in Proposition 0.10. ☐

0.C. **Proof of Theorem 0.1.** Now we are ready to prove Theorem 0.1. We will only deal with the cases which are affected by Lemma 0.2. The proof is based on a discussion with Masaru Nagaoka.

For case (1), denote by R_1 and R_2 the extremal rays of $\overline{\mathrm{NE}}(X)$ and, without loss of generality, we shall assume that $A \cdot R_1 > 0$. Then we have the following diagram



where σ is the contraction corresponding to R_1. The case affected by [Liu20, Lemma 3.2] is that σ is a conic bundle and the induced morphism $A \to Y$ is not an isomorphism. Note that Y is always isomorphic to the projective space $\mathbb{P}^{n-1}$ by [HM99, Main Theorem]. Thus Proposition 0.3 shows that A is isomorphic to a quadric hypersurface. Then, by adjunction formula, we have

$$-K_X = A + (n-1)H_X,$$

where H_X is the pull-back of a hyperplane section of $Y = \mathbb{P}^{n-1}$. Consider the following short exact sequence

$$0 \to \mathscr{O}_X(-A - K_X) \to \mathscr{O}_X(-K_X) \to \mathscr{O}_A(-K_X) \to 0.$$

Tensoring it with $\mathscr{O}_X((d-n+1)H_X)$ yields

$$0 \to \mathscr{O}_X(dH_X) \to \mathscr{O}_X(-K_X - (n-1-d)H_X) \to \mathscr{O}_A(-K_A - (n-1)H_X) \to 0.$$

Here we use the fact that $\mathscr{N}_{A/X} \cong \mathscr{O}_A(-dH_X)$. Moreover, as $\mathscr{O}_A(-K_A) \cong \mathscr{O}_A(n-1)$, pushing-forward the exact sequence by σ yields

$$0 \to \mathscr{O}_{\mathbb{P}^{n-1}}(d) \to \mathscr{E} \to \mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(-1) \to 0.$$

This implies

$$\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(d) \oplus \mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(-1).$$

Note that $-K_X - (n-1-d)H_X$ is σ-very ample, it follows that X is embedded in $\mathbb{P}(\mathscr{E})$ as a divisor such that $X \in |2\zeta + aH|$ for some integer a, where H is the pull-back of a hyperplane section of $\mathbb{P}^{n-1}$ to $\mathbb{P}(\mathscr{E})$ and ζ is the tautological divisor of $\mathbb{P}(\mathscr{E})$. Then we obtain

$$\begin{aligned} -\zeta|_X - (n-1-d)H_X = K_X &= (K_{\mathbb{P}(E)} + 2\zeta + aH)|_X \\ &= -\zeta|_X + (d-1-n+a)H_X. \end{aligned}$$

Here we use the fact that $\zeta|_X = -K_X - (n-1-d)H_X$. Hence we have $a = 2$. Moreover, let $F \subset \mathbb{P}(\mathscr{E})$ be the prime divisor corresponding to the quotient

$$\mathscr{E} \to \mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(-1).$$

Then A is contained $X \cap F$ and we have

$$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(F)|_X \cong \mathscr{O}_X(\zeta - dH) \cong \mathscr{O}_X(A).$$

Hence, we obtain $A = X \cap F$ and we are in case (1.3).

For case (2), there exists a blow-up $\sigma : X \to Y$ along a smooth centre of codimension 2, Y is a smooth Fano variety and $A \cdot R > 0$, where R is the extremal ray of $\overline{\mathrm{NE}}(X)$ generated by the class of a curve contracted by σ. Moreover, there exists a Fano manifold Z of dimension $n-1$, $\rho(Z)=1$ and a $\mathbb{P}^1$-bundle $\pi : Y \to Z$. Set $A_Y = \sigma(A)$. Then $A \to A_Y$ is an isomorphism and C is contained A_Y. Denote by d' the unique positive integer such that $C \in |\mathscr{O}_{A_Y}(d')|$. The case affected by [Liu20, Lemma 3.2] is that A_Y is not a nef divisor in Y. Then the pair (Y, A_Y) is isomorphic to one of the varieties listed in (1.1) and (1.3). The case (1.1) is already done and it remains to consider the case (1.3). Nevertheless, in this case, since $Y \to Z$ is a $\mathbb{P}^1$-bundle and there exists a contraction $Y \to Z'$ sending A_Y to a point, by [CD15, Lemma 3.9], the divisor A_Y must be a section of $Y \to Z$, which is a contraction. Hence, the case (1.3) does not happen.

0.D. **Some other typos.** In [Liu20, Proposition 2.10], the condition "$\mathscr{L}|_D$ is very ample" should be replaced by the condition "$\mathscr{L}|_D$ is simply generated". Similarly, in [Liu20, Proposition 2.11], the condition "$\mathscr{L}|_Y$ is very ample" should be replaced by the condition "$\mathscr{L}|_Y$ is simply generated". In the proof, these two propositions are used in the case with D and Y being a rational homogeneous space of Picard number 1 and it is known that any ample line bundles on rational homogeneous spaces are simply generated (see for instance [RR85, Theorem 1]).

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JIE LIU, INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA
Email address: jliu@amss.ac.cn